University of North Georgia

Department of Mathematics

Instructor: Berhanu Kidane

Course: College Algebra Math 1111

Text Book: For this course we use the free e – book by Stitz and Zeager with link:

http://www.stitz-zeager.com/szca07042013.pdf

Other online resources:

e – book: http://msenux.redwoods.edu/IntAlgText/

Tutorials: http://www.wtamu.edu/academic/anns/mps/math/mathlab/col_algebra/index.htm

For more free supportive educational resources consult the syllabus

Chapter 6
Exponential and logarithmic Functions (Page 417)

Objectives: By the end of this chapter students should be able to:

- Identify Exponential and logarithmic Functions
- Identify graphs of exponential and logarithmic functions
- Sketches graphs of Exponential and Logarithmic functions
- Identify the relationship between exponential and logarithmic functions
- Identify and state rules of exponential and logarithmic functions
- Find domain and range of exponential and logarithmic functions
- Simplify exponential and logarithmic functions using their rules

Motivation

1) Interest: Compound

Compounded Continuously

Formulas:

$$A = P\left(1 + \frac{r}{n}\right)^{nt} \quad \textbf{(Compound Interest)}$$

$$A = Pe^{rt} \quad \textbf{(Continuous Compounding)}$$

A = Amount
P = Principal
r = Rate of interest (in %)
t = Time (usually in years)
n = Number of times amount is compounded

2) Radioactive Decay & Population Growth

Radioactive Decay: If m_0 is the initial mass of a radio active substance with half life **h**, then the mass $m(t)$ remaining at time **t** is modeled by the function

$$m(t) = m_0 e^{-rt}, \text{ where } r = \frac{\ln 2}{h}$$

Population Growth: A population that experiences a population growth increases according to the model: $n(t) = n_0 e^{rt}$

where $n(t)$ = Population at time **t**, n_0= Initial size of population, **r** = relative rate of growth (expressed as a proportion of the population), **t** = time.

Example: **C-14 Dating**. The burial cloth of an Egyptian mummy is examined to contain 59% of the C-14 it contained originally. How long ago was the mummy buried? (The half-life of C-14 is 5730 years)

Example: **World Population**. The population of the world was 5.7 billion in 1995, and the observed relative growth was 2% per year.

a) By what year will the population have doubled?
b) By what year will the population have tripled?

Compound Interest

Compound Interest is calculated by the formula:

$$A(t) = P\left(1 + \frac{r}{n}\right)^{n\,t}$$

Example 4: If $4000 is borrowed at a rate of 5.75% interest per year, compounded quarterly, find the amount due at the end of the given number of years. a) 4 years b) 6 years c) 8 years

For $r = 1$, the compound interest formula becomes $A(t) = P\left(1 + \frac{1}{n}\right)^{nt}$.

The Number e

Consider the expression $\left(1 + \frac{1}{n}\right)^{n}$. We would like to investigate the value that this expression gets close to if n keeps getting larger. That is as $n \to \infty$, $\left(1 + \frac{1}{n}\right)^{n} \to$?

n	$\left(1 + \frac{1}{n}\right)^{n}$
1	2
10	2.593742
100	2.7048138
10000	2.71814592
100000	2.718268273
1000000	2.7182804693
10000000	2.718281692544
10^8	2.7182818148676
10^9	2.71828182709990
. . .	
∞	**2.71828182845904…**

From the **above table** we can make the following observation:

As n increases without bound $\left(1 + \frac{1}{n}\right)^{n}$ approaches the number e, or equivalently

When $n \to \infty$ the value $\left(1 + \frac{1}{n}\right)^{n} \to e$

6.1 Exponential Functions

Exponential Functions of base a

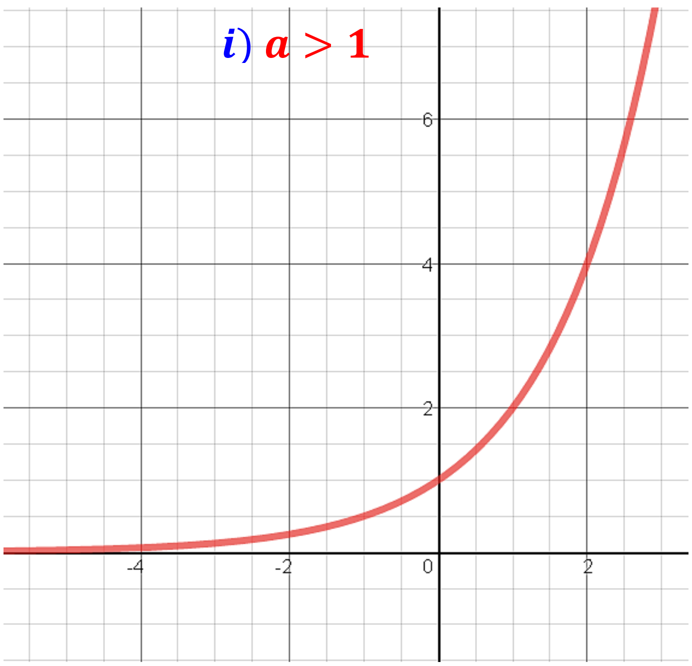
Definition: An **exponential function** with **base a** is the function defined by $f(x) = a^x$, where $a > 0$ and $a \neq 1$.

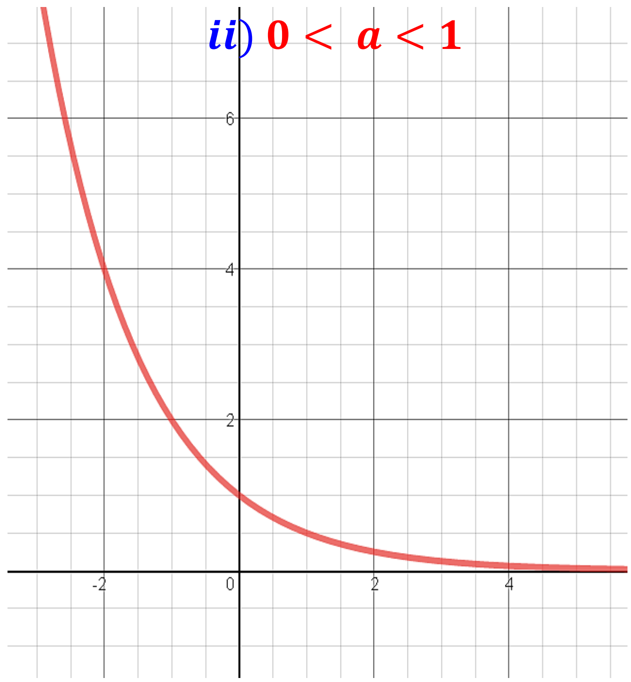
Example 1: a) $f(x) = 2^x$

b) $g(x) = \left(\frac{1}{2}\right)^x = 2^{-x}$

c) $f(x) = e^x$

Graphs of $f(x) = a^x$: there are **two cases i) $a > 1$ and ii) $0 < a < 1$**





Properties of the exponential function $f(x) = a^x$:

1) The domain of $f(x) = a^x$ is the set of all real numbers = $(-\infty, \infty)$

2) The function $f(x) = a^x$ is increasing for $a > 1$ and decreasing for $0 < a < 1$

3) The range of $f(x) = a^x$ is $\{ y \mid y > 0 \} = (0, \infty)$

4) The function $f(x) = a^x$ has y intercept $(0, 1)$ but has **no** x - intercept

5) The function $f(x) = a^x$ is a **one - to – one function**, hence it is invertible.

Example 2: Sketch the graph of the following exponential functions:

a. $f(x) = 2^x$
a. $f(x) = 0.8^x$
b. $f(x) = \sqrt[3]{3}^x$
d) $f(x) = \left(\frac{1}{2}\right)^x$
e) $f(x) = 3^x$
f) $f(x) = 0.6^x$

Transformations:

Translations, **Reflections,** and **Vertical** and **Horizontal Stretches** and **Shrinks**

Translations:

1) **Vertical Translation**: $y = f(x) \pm c, for\ c > 0$

The graph of $y = f(x) + c$ is the graph of $y = f(x)$ shifted vertically **c** units up
The graph of $y = f(x) + c$ is the graph of $y = f(x)$ shifted vertically **c** units down

2) **Horizontal Translations:** $y = f(x \pm c), for\ c > 0$

The graph of $y = f(x - c)$ is the graph of $y = f(x)$ shifted horizontally **c** units to the right
The graph of $y = f(x - c)$ is the graph of $y = f(x)$ shifted horizontally **c** units to the left.

Reflections

1) **Across the x-axis:**

The graph of $y = -f(x)$ is the **reflection** of the graph of $y = f(x)$ across the **x-axis**.

2) **Across the y-axis:**

The graph of $y = f(-x)$ is the **reflection** of the graph of $y = f(x)$ across the **y-axis**.

Stretches and Shrinks

Vertical Stretching and shrinking

To graph $y = cf(x)$:

If $c > 1$, **stretch** the graph of $y = f(x)$ **vertically** by a **factor of c**
If $0 < c < 1$, **shrink** the graph of $y = f(x)$ **vertically** by a **factor of c**

Horizontal Stretching and shrinking

To graph $y = f(cx)$:

If $c > 1$, **shrink** the graph of $y = f(x)$ **horizontally** by a **factor of** $1/c$
If $0 < c < 1$, **stretch** the graph of $y = f(x)$ **horizontally** by a **factor of** $1/c$

Example 3: Sketch the graph (Transformations of Exponential Functions)

a. $f(x) = -2^x$
b. $f(x) = 2^x + 2$
c. $f(x) = 2^{x-1}$
d. $f(x) = -2^{x+1} - 2$

OER West Texas A&M University Tutorial 42: Exponential Functions

The Natural Exponential Function

Definition: The Natural Exponential Function is defined by $f(x) = e^x$, with base **e**.

Continuously Compounded Interest

Example 1: Continuously Compounded Interest is calculated by the formula:

$$A(t) = Pe^{rt}$$

Where $A(t)$ = Amount after t years, **P** = Principal, **r** = Interest rate per year, and **t** = Number of years

Example 2: A sum of $5000 is invested at an interest rate of 9% per year compounded continuously

a) Find the value of **A(t)** of the investment after t years

b) Draw a graph of **A(t)**

Laws of Exponents

Laws	Examples
$x^1 = x$	$6^1 = 6$
$x^0 = 1$	$7^0 = 1$
$x^{-1} = 1/x$	$4^{-1} = 1/4$
$x^m x^n = x^{m+n}$	$x^2 x^3 = x^{2+3} = x^5$
$x^m/x^n = x^{m-n}$	$x^6/x^2 = x^{6-2} = x^4$
$(x^m)^n = x^{mn}$	$(x^2)^3 = x^{2\times 3} = x^6$
$(xy)^n = x^n y^n$	$(xy)^3 = x^3 y^3$
$(x/y)^n = x^n/y^n$	$(x/y)^2 = x^2 / y^2$
$x^{-n} = 1/x^n$	$x^{-3} = 1/x^3$

And the Laws about Fractional Exponents:

Laws	Examples
$x^{1/n} = \sqrt[n]{x}$	$x^{1/3} = \sqrt[3]{x}$
$x^{\frac{m}{n}} = \sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m$	$x^{\frac{2}{3}} = \sqrt[3]{x^2} = \left(\sqrt[3]{x}\right)^2$

Proof of the law: $x^{\frac{m}{n}} = \sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m$ follows from the fact that

$$\frac{m}{n} = m \times (1/n) = (1/n) \times m$$

OER West Texas A&M University Tutorial 2: Integer Exponents **Tutorial** 5: Rational Exponents

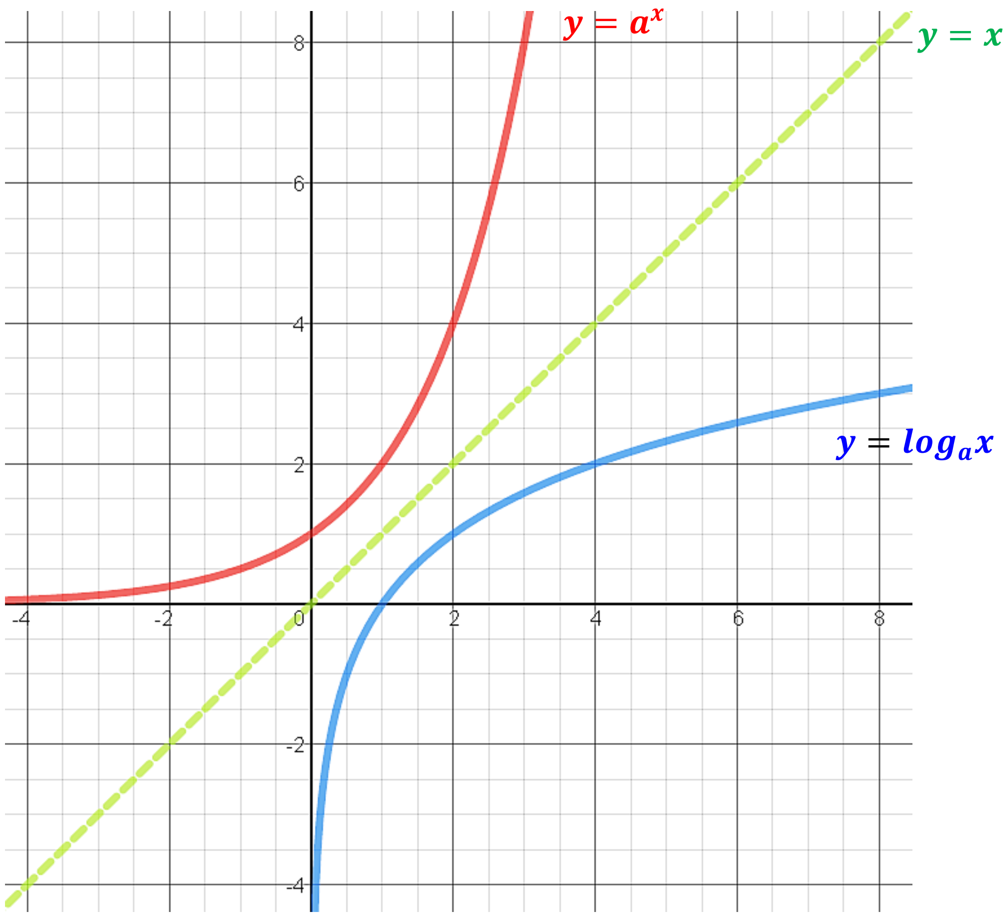
6.2 Logarithmic Functions and Their Graphs **(page 423)**

Consider the exponential function $y = a^x, a > 0$ and $a \neq 1$

- $y = a^x$ **is a one-to-one function, thus it has an inverse**
- **The inverse of** $y = a^x$ **is a function called the logarithmic function**

Recall, the inverse of a function is obtained by interchanging the x and the y in the equation defining the function. Thus, the inverse of $y = a^x$ is given by $x = a^y$ which is the same as $y = log_2 x$. That is we are saying $x = a^y \Leftrightarrow y = log_a x$

Graphically: The graph of $y = log_a x$ obtained by reflecting the graph of $y = a^x$ across the line $y = x$.



Logarithmic Function, Base a

Definition: (log function to any base a**)**

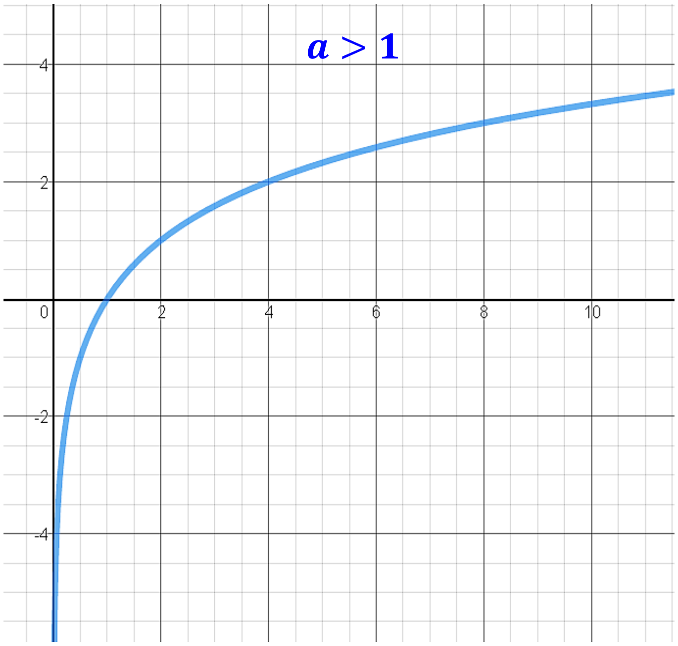
$y = log_a x$ is the number y such that $x = a^y$, where $x > 0$ and $a > 0$ *and* $\neq 1$ is

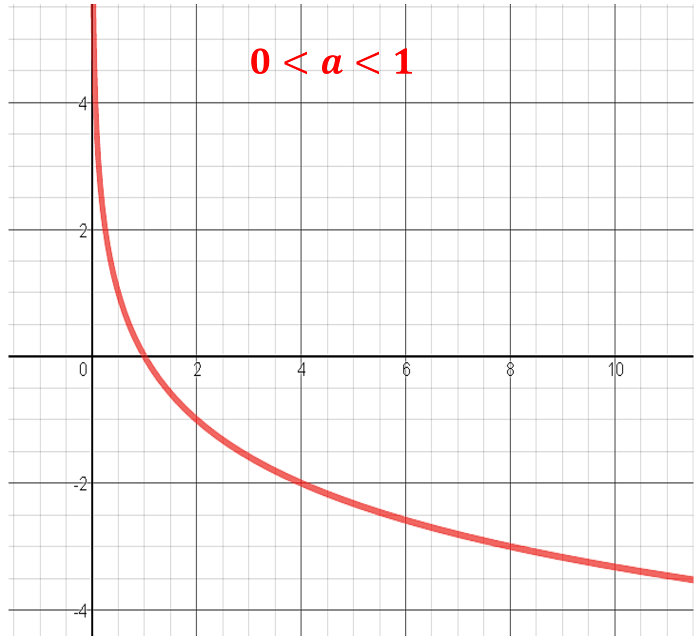
Examples

a) **Case,** $a > 1$: $y = log_2 x$, $y = log_3 x$, $y = log_{1.3} x$; $y = log\ x$; $y = ln\ x$

b) **Case,** $0 < a < 1$: $y = log_{1/2} x$, $y = log_{1/3} x$, $y = log_{0.4} x$; $y = log_{1/7} x$

Graphs of $y = log_a x$: **Two cases i)** $a > 1$ and **ii)** $0 < a < 1$





Properties of the logarithm function $f(x) = log_a x$

1) The domain of $f(x) = log_a x$ is $\{x \mid x > 0\} = (0, \ \infty)$
2) The function $f(x) = log_a x$ is increasing for $a > 1$ and decreasing for $0 < a < 1$
3) The range of $f(x) = log_a x$ is the set of all real numbers, in interval form $(-\infty, \ \infty)$
4) The function $f(x) = log_a x$ has x intercept $(1, \ 0)$ has **no y** - intercept
5) The function $f(x) = log_a x$ is a **one - to – one function**, hence it is invertible.
6) The function $f(x) = log_a x$ is the inverse of the exponential function $y = a^x$ and vice versa

Example 1: Find graph the following logarithmic functions

a) $y = log_3 x$, $y = log_{1.3} x$; $y = log\ x$; $y = ln\ x$

b) $y = log_{1/3} x$, $y = log_{0.4} x$; $y = log_{1/7} x$

OER West Texas A&M University Tutorial 43: Logarithmic Functions

Example 2: Find the domain and graph the following logarithmic functions

a) $y = -log_3 x$ **c)** $y = -log_{0.2}(x+1) + 2$

b) $y = log_2(x-2)$

Example 2: Example 6.1.4. Page 425: Find the domain of the following functions

a) $f(x) = 2\log(3-x) - 1$

b) $g(x) = \ln\left(\frac{1}{x-1}\right)$

Homework page 429: #1 – 74 (odd numbers)

Natural and Common Logarithms

Definition: **1)** Logarithms with **base e** are called **natural logarithms,**

Notation: $ln\ x$ used instead of $log_e x$

2) Logarithms with **base 10** are called **common logarithms**

Notation: $log\ x$ used instead of $log_{10} x$

The **calculator** log is **base 10,** and the calculator ln is **base e.**

Example 3: Find using a calculator:

a) $\log 13$

b) $\log 10$

c) $\ln 9$

d) $\ln e$

e) $log\ 5$

f) $ln\ 5$

Conversion between Exponential and Logarithmic Equations

Exponent Form		Logarithmic Form
$b^y = x$	$\Leftrightarrow$	$y = log_b x$
$e^y = x$	$\Leftrightarrow$	$y = ln\ x$
$10^y = x$	$\Leftrightarrow$	$y = log\ x$

Example 4: Example 6.1.3 Page 424: Reading

Examples 4: Convert to the exponential form

a) $\log 1000 = 3$

b) $\log_3 81 = 4$

c) $\log 5 = b$

c) $\ln e = 1$

d) $\ln \sqrt[3]{e} = 1/3$

f) $\ln 9 = t$

Example 5: Convert each of the following to a **logarithmic** or **exponential** equation*:*

a) $16 = 2^x$

b) $\log_2 32 = 5$

c) $\log_3 9 = 2$

d) $7^2 = 49$

e) $10^{-3} = 0.001$

f) $x = \log_t M$

g) $\ln 4 = y$

h) $27^{1/3} = 3$

Properties of Logarithms (page 437)

OER West Texas A&M University Tutorial 44: Logarithmic Properties

1) $log_b(xy) = log_b x + log_b y$ **(Product Rule)**

2) $log_b(x/y) = log_b x - log_b y$ **(Quotient Rule)**

3) $log_b x^P = P \times log_b x$ **(Power Rule)**

4) $log_b x = \frac{\log_c x}{\log_c b}$, $for\ c > 0\ and\ c \neq 1$ **(Change of Base)**

If we change the base b to $c = 10$ *or* $c = e$, then the **change** of **base** formula becomes:

$$log_b x = \frac{\log x}{\log b} \quad \textbf{OR} \quad log_b x = \frac{\ln x}{\ln b}$$

5) **Other properties: Let $b > 0$ and $b \neq 1$, then:**

a) $log_b 1 = 0$, and so $\ln 1 = 0$

b) $log_b b = 1$, and so $\ln e = 1$

c) $log_b b^x = x$, and so $\ln e^x = x$

d) $b^{log_b x} = x$, and so $e^{\ln x} = x$

Example 1: Example 6.2.1 page 438: Reading

Example 1: Find each of the following using properties of log.

a) $\log 10000$

b) $log_2\left(\frac{1}{8}\right)$

c) $\log_5 5^3$

d) $\log_7 49$

e) $\log 100$

f) $\log_3 3$

Example 2: Find the values of the following using log properties

a) $\log_{10} 5$

b) $\log_{1/3} 81$

c) $\log \sqrt[3]{42}$

Example 3: Simplify the following

a) $\left(2^{\sqrt{5}}\right)^{\sqrt{20}}$

b) $\log_2(\log_9 81)$

c) $\log_2(128/16)$

d) $e^{\ln \sqrt[3]{81}}$

Example 4: Evaluate without a calculator whenever possible otherwise use calculator

a) $\log \sqrt[3]{100}$

b) $log_3 \sqrt[4]{27}$

c) $log_2 25$

d) $ln(\sqrt[7]{e^2})$

Example 5: Evaluate:

a) $\log_2 5$

b) $\log_{0.32} 99$

Example 6: **Write as a single log:**

a) $log_2(x-2)+3log_2x-log_2(3+x)$ c) $2\log_4 x+\log_4 y-\frac{1}{3}z$

b) $\log_b x+2\log_b y-3\log_b x$

Example 7: **Expand** using **log properties:**

a) $\log(3\sqrt{x})$ d) $\log_b(x^2y^3z^2)$

b) $log_5\left(\frac{\sqrt{x+1}}{9x^2(x-3)}\right)$ e) $log_a\left(\sqrt[3]{\frac{a^2b}{c^4}}\right)$

c) $\log\left(\frac{x^{1/2}}{y^2\sqrt[3]{z}}\right)$

Homework page 445: #1 – 42 (odd numbers)

Solving Exponential and Equations Log Equations:

OER West Texas A&M University: Tutorial 45: Exponential Equations;

Tutorial 46: Logarithmic Equations

Form	**Strategy**
1. $b^x=b^y$	**Bases** are the **same**, **drop bases** to obtain $x=y$
2. $b^x=y$	**Take** *log* or *ln* of **both sides** to change to the *log* form
3. $log_bx=log_by$	**Bases** are the **same**, **drop** the *logs* to obtain $x=y$
4. $log_bx=y$	**Convert** to **exponential form** to solve $b^y=x$

Example1: **Solve each of the following**

a) $4^{3x}=32^{x-2}$ g) $4^{x+3}=3^{-x}$

b) $e^{x+3}=e^{x^2-4x}$ h) $7e^{x+3}=5$

c) $2^{5x}=64$ i) $3^x-3^{-x}=4$

d) $9^{x^2}.3^{5x}=27$ j) $2e^{4x}+5e^{2x}+3=0$

e) $3^{x^2-5x}=\frac{1}{81}$ f) $3^x=7$

Example 2: State the domain and solve the following

a) $\log_2 x=6$ e) $log_4x+log_4(x+1)=log_42$

b) $log_3x+log_3(2x-3)=3$ f) $\log(x+2)-3\log 2=1$

c) $log_3x+log_3(x+1)=log_32$ g) $\log_b 81=-2$

d) $log_2(x+1)+log_2(3x-5)=log_2(5x-3)+2$

Homework page 456: #1 – 33 (odd numbers)
Homework page 466: #1 – 24 (odd numbers)