University of North Georgia

Department of Mathematics

Instructor: Berhanu Kidane
Course: College Algebra Math 1111
Text Book: For this course we use the free e – book by Stitz and Zeager with link:
http://www.stitz-zeager.com/szca07042013.pdf
Other online resources:
e – book: http://msenux.redwoods.edu/IntAlgText/
Tutorials: http://www.wtamu.edu/academic/anns/mps/math/mathlab/col_algebra/index.htm

For more free supportive educational resources consult the syllabus

Functions and Relations (Page 20)

Objectives: By the end of this chapter students should be able to:

- **Identify relations**
- **Define a relation**
- **Define a function and find the domain and range of a function**
- **Identify graphs of functions and sketch graphs of functions**
- **Describe the different transformations of functions, and sketch graphs using transformation of functions**
- **Identify one – to – one functions**

Introduction to Relations and Their Graphs (Page 20 – 28)

Relations

Important Ideas **Set** and **Ordered Pairs**

Definition (A set): A set is a collection of well-ordered objects.

Examples: a) The set of students in this class

b) The set of natural less than 11 $= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

c) $\boldsymbol{E}$ $=$ The set of even natural numbers

$= \{2, 4, 6, 8, 10, . \;.\; .\}$

$= \{\, x: x \text{ is an even natural number} \,\}$

Definition (Ordered Pairs):

A pair which is written in the form $(\boldsymbol{a}, \boldsymbol{b})$ is called an **ordered** pair. In the pair $(\boldsymbol{a}, \boldsymbol{b})$, $\boldsymbol{a}$ is called **first** or $\boldsymbol{x}$ **coordinate** (or entry) and **b** is called **second** or **y** **coordinate** (or entry).

Note: $(\boldsymbol{a}, \boldsymbol{b}) = (\boldsymbol{c}, \boldsymbol{d})$ if and only if $\boldsymbol{a} = \boldsymbol{c}$ and $\boldsymbol{b} = \boldsymbol{d}$.

Definition (Relation)**:** A relation is a set of ordered pairs.

Example 1:

a) $\boldsymbol{R} = \{\, (\mathbf{2}, -\mathbf{5}),\ (\mathbf{5},\ \mathbf{6}),\ (-\mathbf{7}, \mathbf{6}),\ (\mathbf{2}, \mathbf{7}),\ (-\mathbf{7},\ \mathbf{3})\}$

b) $\boldsymbol{F} = \{\, (\boldsymbol{a}, \boldsymbol{b}),\ (\mathbf{3},\ \mathbf{4}), (\boldsymbol{c}, \boldsymbol{d}) \,\}$

Example 2:

a) $\boldsymbol{R} = \{\, (\boldsymbol{x}, \boldsymbol{y}) : \ \boldsymbol{x} + \boldsymbol{y} > \mathbf{0},\ \boldsymbol{x} - \boldsymbol{y} < \mathbf{1},\ \boldsymbol{y} \le \mathbf{2}\}$

b) $\boldsymbol{F} = \{(\boldsymbol{x}, \boldsymbol{y}): \boldsymbol{y} > \boldsymbol{x}^{\mathbf{2}} \text{ and } \boldsymbol{y} \le \mathbf{4}\}$

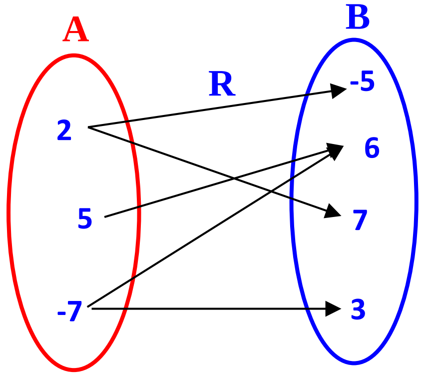
A **relation** can also be **represented** by:

i) A Venn-diagram, ii) A graph, or iii) An Equation

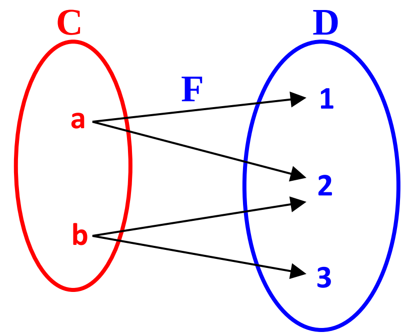
i) Venn-diagrams of relations

Example 1:

a)



b)



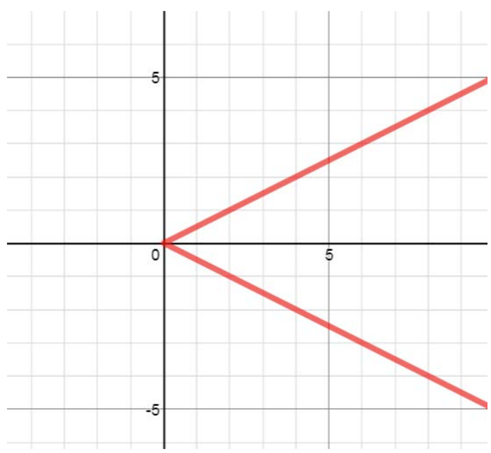
R is a relation from **A** into **B**

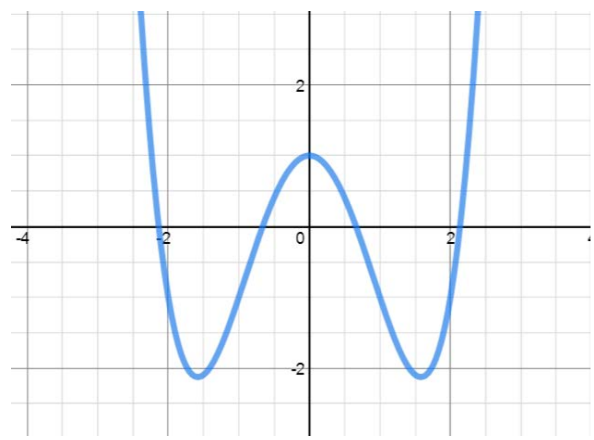
Using the set notation **R** $=$

F is a relation from **C** into **D**

Using the set notation **F** $=$

ii) Graphs of relations





iii) Equations defining relations

Example 2:

a) $\boldsymbol{y = x^2 - 5x + 9}$

b) $\boldsymbol{|\,y\,| = x + 1}$

c) $\boldsymbol{y \geq -2x - 4, y > x + 1}$ *and* $\boldsymbol{y \leq 2}$

d) $\boldsymbol{2x - 6y > 3}$ *and* $\boldsymbol{x + y < 1}$

e) $\boldsymbol{y^2 = x}$

f) $\boldsymbol{x^2 + y^2 = 1}$

Homework

Page 29 – 32: 1-56 odd numbers (Stitz and Zeager Book)

Functions and Their Graphs (page 43)

Important Ideas:

Definition of a Function, Venn diagrams, Graphs,
Domain, Range, Functional Values,
Functional Notations, Equations Defining Functions,
Vertical Line Test

Definition 1: A **function** is a **relation** for which **each element** in the **domain corresponds** to **exactly one element** in the **range**. In other words, **every** x **can only** be **paired with one** $\mathbf{y}$.

Or Equivalently

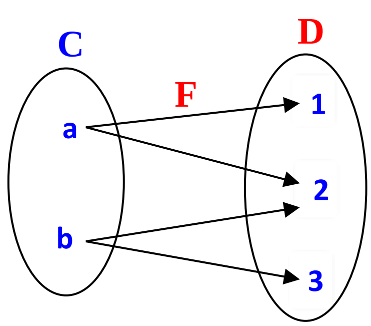
Definition 2: A **relationship** between **two variables**, typically x and $\mathbf{y}$, is called a **function** if there is a rule that assigns to **each value** of x one and only one value of $\mathbf{y}$. We then say that $\mathbf{y}$ is a **function** of x.

Note: For functions **the same** x **value cannot** have **2 different** $\mathbf{y}$ **values**.

Example 1: a) $\mathbf{R} = \{(\mathbf{2},\mathbf{3}),\ (\mathbf{1},\mathbf{5}),\ (\mathbf{0},\mathbf{4})\}$, is a **function**, why?

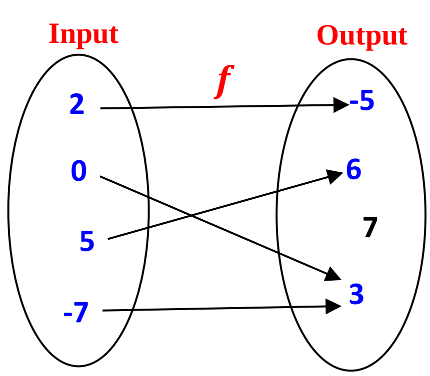
b) $\mathbf{R} = \{(\mathbf{2},\mathbf{3}),\ (\mathbf{2},\mathbf{5}),\ (\mathbf{0},\mathbf{4}),\ (\mathbf{3},\mathbf{4})\}$, is **not** a **function**, why?

Example 2: Venn diagrams



F is not a function, why?

Example 3:



f is a **function,** every x is **paired** to **exactly** one $\mathbf{y}$**.**

Domain and Range

Definition: Let f be a function.

- The **set** of all **first entries** is called the **DOMAIN** of the function f.
- The **set** of all **second entries** is called the **RANGE** of the function f.

Examples 4: Domain and range

a) $f = \{ (1,1),\ (-1,1),\ (2,4),\ (3,\ 9) \}$

Domain of $f = \{1,\ -1,\ 2,\ 3\}$, **Range of** $f = \{1,\ 4,\ 9\}$

b) $g = \{ (1,\ 4),\ (2,4),\ (3,5), (6,10) \}$

Domain of g = **Range of** g =

Homework
Page 49 – 51: 1-50 odd numbers (Stitz and Zeager Book)

The Values of a Function and Functional Notation

By the **value of a function** we mean the **value of y**.

Functions are often denoted by the letters ***f, F, g*** *and* ***G*** etc.

Note: If f is a function, then for each **number** x in its **domain** the corresponding **image** in the **range** is **designated** by the symbol $f(x)$ and read as " $f\ of\ x$" or as "$f\ at\ x$". We **refer** to $f(x)$ as **the value** of $f\ at\ x$, *or* the **output corresponding** to x, or the **image of** x. **Note that** $f(x)$ **does not mean** $f\ times\ x$.

Note in functions:

Inputs or Pre-images are **1st** or $x - entries$, **1st** or $x - coordinates$ or $x - values$

Outputs or Images are **2nd** or $y - entries$, **2nd** or $y - coordinates$ or $y - values$

Example 1: *Read* each symbol.

a) $g(x)$; "g of x" b) $f(2)$; "f of 2" c) $g(-1)$; "g of -1"

d) $f(x^2 - 1)$; "f of $x^2 - 1$" e) $f(g(x))$; "f of g of x"

Functional Notation (page 61)

Examples: Let $y = x^2 + 1$; write $f(x)$ for the value **y.**

Then we call $f(x) = x^2 + 1$ the functional notation for $y = x^2 + 1$

Similarly: $f(x) = 3x - 5$ is a functional notation for $y = 3x - 5$; and
$f(x) = x^2$ is a **functional notation** for $y = x^2$ **and so on.**

Example 2: Finding functional values

Let $f(x) = 2x + 1$. **What is** $f(3)$, $f(-5)$, **and** $f(A)$**?**

Solution: Note, f maps x **to 2 times** x **plus 1**

$$f(3) = 2(3) + 1 = 7$$

$$f(-5) = 2(-5) + 1 = -9$$

$$f(A) = 2(A) + 1 = 2A + 1$$

Example 3: Let $y = 1 - x^3$. What is the value of y when

a) $x = 0$? b) $x = -1$?

c) $x = q$? d) $x = -q$?

Example 4: If $h(x) = -2x + 1$, then

a) $h(x^3) =$

b) $h(x + 5) =$

c) $h(10) =$

Example 5: For $f(x) = x^2 + 3x + 1$, evaluate the following:

a) $f(0)$ b) $f(2)$

c) $f(-x)$ d) $f(x + 1)$

Homework
Page 63 – 68: 1-62 odd numbers, 63 – 68 odd numbers (Stitz and Zeager Book)

Graphs of Functions

The Graph of a Function

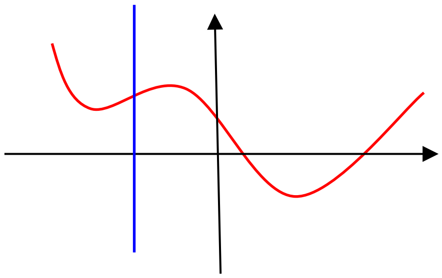
If f is a function with domain A, then the graph of f is the set of ordered pairs $\{(x, f(x)) \mid x \in A\}$ **plotted** in the **coordinate plane**. In other word the graph of f is the graph of the equation $y = f(x)$.

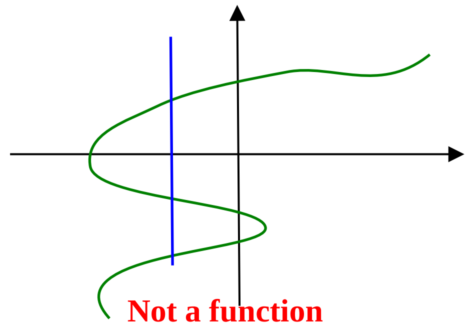
Example 1: Sketch the graph of the following functions

a) $y = x^2$ b) $y = 2x + 1$ c) $f(x) = x^3$

Vertical Line Test

A set of points in the xy - plane is the **graph of a function** if and only if **every vertical line intersects** the graph **in at most one point**.





Equations Defining Functions

To **determine** whether an **equation** in x and **y** **defines** a **function**, **solve** the **equation** for **y**. If we get only **one equation** expressed in terms of x, then the original equation is a function

Examples:

a) Show that $2y - 4x = 6$ defines a function.

Solve for **y** to get $y = 2x + 3$

The last equation shows that every x is paired to exactly one **y**.

b) Show that $y^2 = x$ is **not** a function**.** S**olving** for *y* **gives** $y = \pm\sqrt{x}$

If $x = 1$, then $y = \pm 1$, that is $x = 1$ **corresponds** to **two y values**

Thus, the equation $y^2 = x$ **does not** define a function.

c) $y = 2x + 1$ is a function of x since each **x-value,** input, results in only **1 y-value**.

d) $|y| = x$ is **not** a function of x**,** since $x = 9$ corresponds to both **y = 9** *and* **y = -9**.

e) $y = x^2$ is a function of x since each x-value, input, results in only 1 y-value, output.

f) Show that $x^2 + y^2 = 1$ is **not** a function

The Zeros of a Function

Definition: Let f be a function if $f(r) = 0$ for number r, then r is called the **zero of** f.

Example 1: Find the zeros of $f(x) = x^2 - 1$

Solution: To find the **zeros** of f we set $f(x) = 0$ and **solve for** *x*

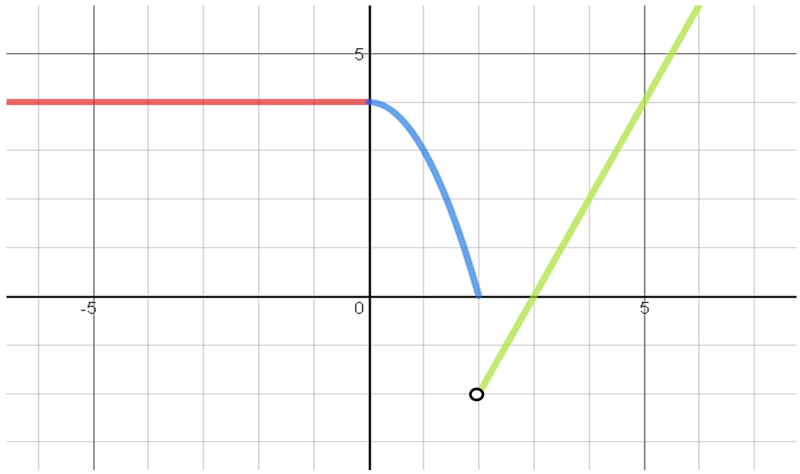
$x^2 - 1 = 0$, which gives $x = -1$ *or* $x = 1$.

Thus **-1** and **1** are the zeros of the function $f(x) = x^2 - 1$.

Piecewise Defined Functions

Piece-wise functions are formed by more than one function. Each function is defined for a specific set of values (intervals).

Example 1: Let $f(x) = \begin{cases} 4, & for\ x \le 0 \\ 4 - x^2, & for\ 0 < x \le 2 \\ 2x - 6, & for\ x > 2 \end{cases}$ Find $f(0),\ f(-2), f(1), and\ f(5)$



Example 2: Amazon charges **\$4** for an order under **\$35** but provides free shipping for orders of **\$35** or more. The cost ***C*** of **an order is a function** of the total **price *x*** of the books purchased, given by

$$C(x) = \begin{cases} x + 4, & if\ x < 35 \\ x, & if\ x \ge 35 \end{cases}$$

Example 3: In a certain state the **maximum speed** permitted on freeway is **65m/h**, and the **minimum** is **40m/h**. The **fine F** for violating these limits is **\$15** for **every mile above** the **maximum** or **below** the **minimum**

a) If ***x*** the **speed** at which you are driving, then the fine function **F** is piecewise and given by the:

$$F(x) = \begin{cases} 15(40 - x), & if\ 0 < x < 40, \\ 0 & if\ 40 \le x \le 65 \\ 15(x - 65) & if\ x > 65, \end{cases}$$

b) Find **F(30), F(50),** and **F(75)**

Example 4: Evaluate each for the following piece-wise function.

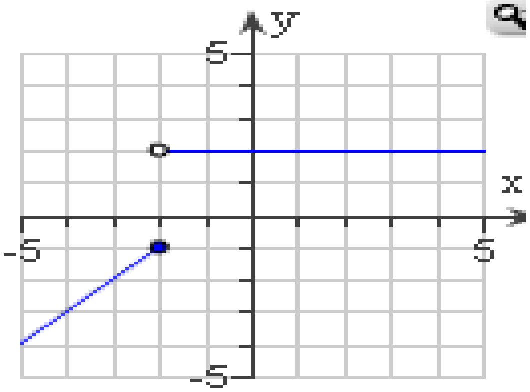
$$f(x) = \begin{cases} x^2 & for\ \ x < -1 \\ x+1 & for\ -1 \le x < 3 \\ 4 & for\ x \ge 3 \end{cases}$$

a) $f(-3) =$ b) $f(0) =$ c) $f(5) =$

Example 5: Let

$$f(x)=\begin{cases} x+1, & if \;\; x \le -2 \\ 2, & if \;\; x > -2 \end{cases}$$

(Graph shown to right)

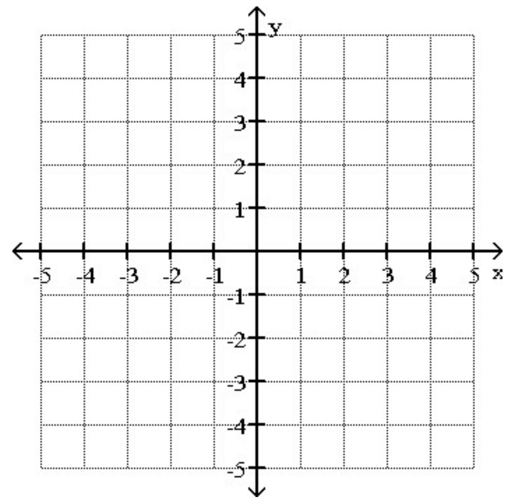


Find:

a) $\boldsymbol{f(-7)} =$
b) $\boldsymbol{f(-2)} =$
c) $\boldsymbol{f(0)} =$
d) $\boldsymbol{f(10)} =$

Example 4: Sketch the graph of the piece-wise function, find the domain, range and find the values of the function at the given points.

$$f(x)=\begin{cases} x+3 & if \quad x \le -2 \\ 3 & if \quad x > -2 \end{cases}$$



a) $\boldsymbol{f(-3)} =$

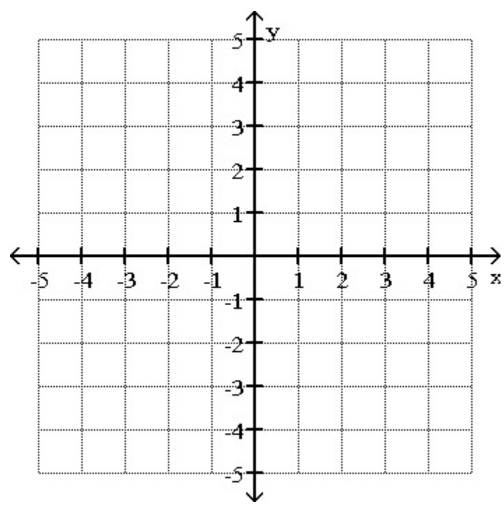
b) $\boldsymbol{f(-2)} =$

c) $\boldsymbol{f(110)} =$

Example 5: Graph the piecewise function ***f***

$$f(x)=\begin{cases} -3-x, & if \quad x \le 2 \\ 2x, & if \quad -2 < x \le 2 \\ x^2-4x+3, & if \quad x > 2 \end{cases}$$

Find



a) $\boldsymbol{f(0)}$

b) $\boldsymbol{f(-2)}$

c) $\boldsymbol{f(10)}$

OER West Texas A&M Tutorial 30: Introduction to Functions

Homework Exercise 1.6.2: page 107 #1 – 20 (Stitz and Zeager Book)

Difference Quotient and Average Rate

Difference Quotient (Page 79)

Let f be a function the **difference quotient** of f is defined as $\frac{f(x+h)-f(x)}{h}$

Example 4: Find the difference quotient, simplify your result.

a) $f(x) = x^2 + 2x + 1$

b) $f(x) = 2x - 3$

c) $f(x) = x^3 + 2x + 1$

d) $f(x) = \frac{x^2+2x-4}{x}$

Example 4: Reading: Example 1.5.2, page 79 – 81;

Solution: $f(x) = x^2 + 2x + 1$

$$a) \quad \frac{f(x+h)-f(x)}{h} = \frac{\left[(x+h)^2+2(x+h)+1\right]-\left[x^2+2x+1\right]}{h}$$

$$= \frac{\left[x^2+xh+xh+h^2+2x+2h+1\right]-\left[x^2+2x+1\right]}{h}$$

$$= \frac{x^2+xh+xh+h^2+2x+2h+1-x^2-2x-1}{h}$$

$$= \frac{2xh+h^2+2h}{h}$$

$$= 2x+h+2$$

Homework: Exercise 1.5.1 page 84: #21 – 45 (Stitz and Zeager Book)

Average Rates of Change

Definition: The average rate of change of $f(x)$ with respect to x for a function f as x changes from a *to* b is defined by $\frac{f(b)-f(a)}{b-a}$

Example: Find the **average rate** of change for the following.

a) $f(x) = x^2 - 2x + 1$ between $x = 0$ and $x = 3$

b) $y = \sqrt{x}$ between $x = 1$ and $x = 4$

c) $y = x^3$ between $x = -2$ and $x = 2$

Basic Graphs

1) A constant function
2) The identity function
3) The absolute value function
4) $y = x^2$; The Square function
5) The square root function
6) The cubic and cube root functions
7) The reciprocal function
8) The greatest integer function

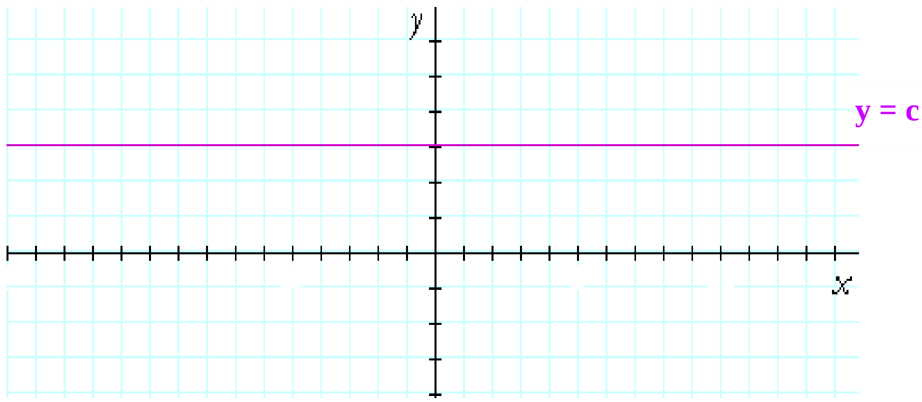
OER http://www.themathpage.com/aprecalc/graph-of-parabola.htm#absolute

1) A Constant Function

A constant function has the general form

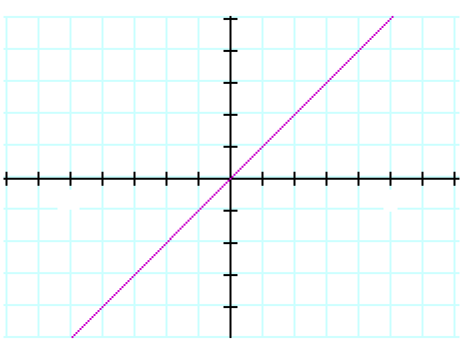
$$y = f(x) = c,$$ where c is a constant, that is, a number

For example of the constant function $y = f(x) = 3$ Its graph is a straight line parallel to the *x*-axis.



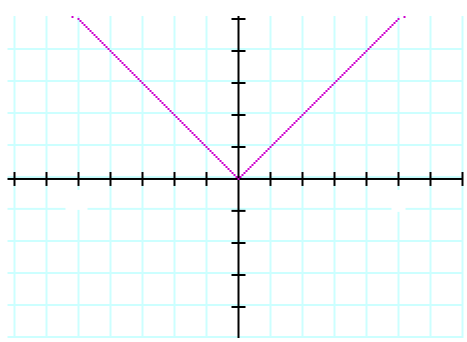
Question: Find the domains and ranges of the constant function

2) The Identity Function and the Absolute Value Function



$y = x$

The identity function

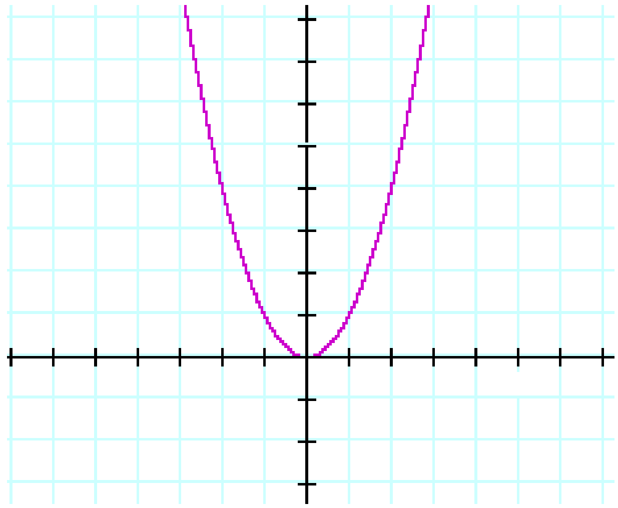


$y = |x|$

The absolute value function

Question: Find the domains and ranges of the identity and the absolute value functions.

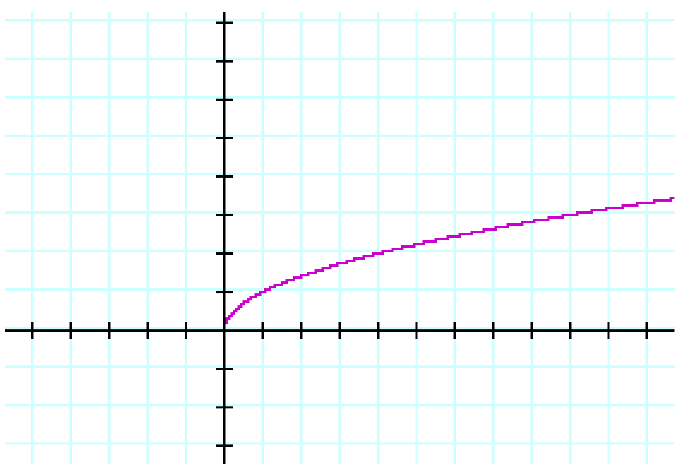
3) A) $y = x^2$ the square function, Parabola



$y = x^2$

A parabola

B) $y = \sqrt{x}$ The Square Root Function



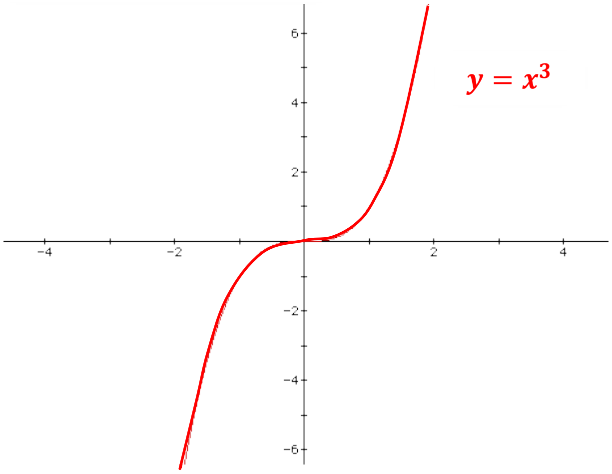
$y = \sqrt{x}$

The square root function

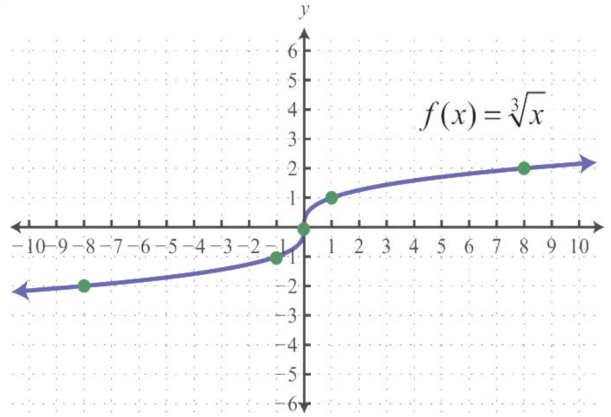
Question: Find the domains and ranges of the Parabola and the square root functions.

4) The Cubic and the Cube Root Functions

The Cubic Function



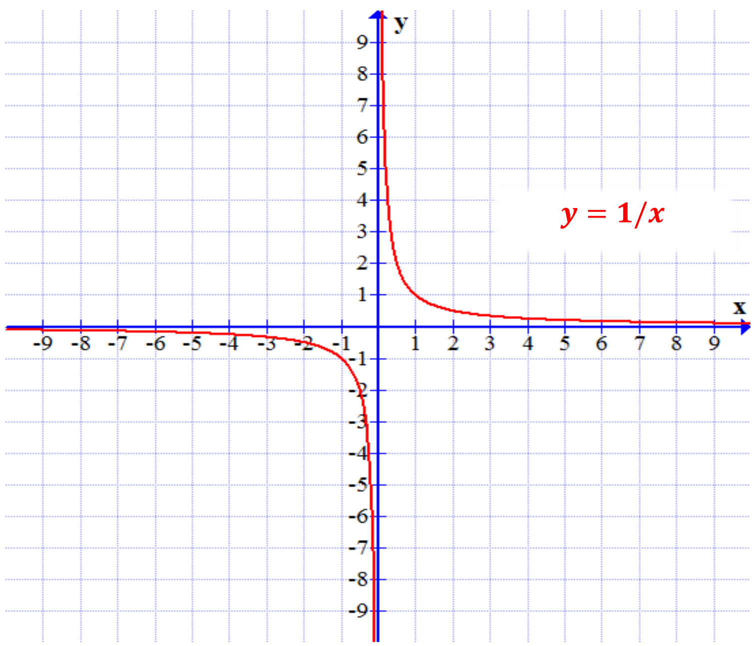
The Cube root Function



Question: Find the domain and range of the cubic and the cube root functions

5) The Reciprocal Function

The reciprocal function $y = f(x) = {}^{1}/_{x}$



Question: Find the domain and range of the cubic functions

6) The Greatest Integer function

The greatest integer function is denoted and defined by $y = f(x) = [x]$

$[x]$ Means the **greatest integer less than** or **equal** to x

Example: Let $f(x) = [x]$. Find the following values.

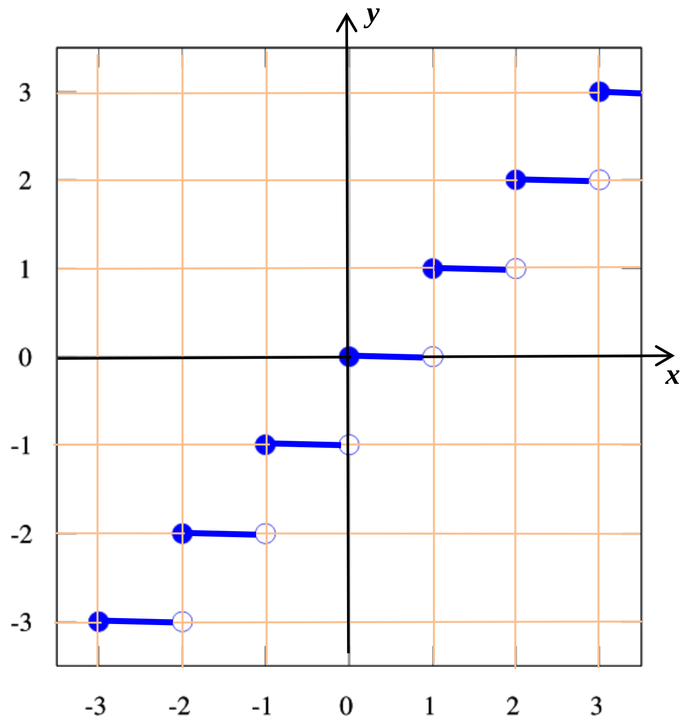
a) $f(0.5)$

b) $f(3.1)$

c) $f(-0.25)$

d) $f(-3)$

Graph of the Greatest Integer Function $y = [x]$



Question: Find the domain and range of the greatest integer functions

OER West Texas A&M Tutorial 31: Graphs of Functions, Part I

More on functions (Page 93)

- Increasing, decreasing and constant functions
- Even Odd Functions and symmetry
- Combination of functions
- Transformation and symmetry

Increasing, decreasing and constant functions

Definition:

a) A function f is said to be an **increasing** function on an interval **I**,

if for all x_1 and x_2 in **I**, $x_1 < x_2$ implies that $f(x_1) > f(x_2)$.

- **Increasing**: where the function is **rising**.

  Trace the graph from left to right; where you go up is where the graph is increasing

b) A function f is said to be an **decreasing** function on an interval **I**,

if for all x_1 and x_2 in **I**, $x_1 < x_2$ implies that $f(x_1) > f(x_2)$.

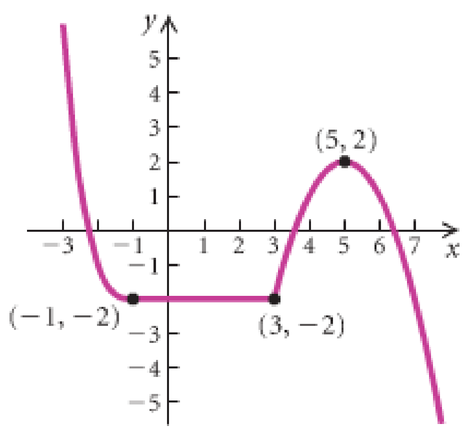
- **Decreasing**: where the function is **falling**.

  Trace the graph from left to right; where you go down is where the graph is decreasing

c) If the value of a function f **does not change** in an interval **I**, then f is **constant** on **I**

- **Constant:** where the function is horizontal

Example 1: Determine the intervals where the graph is increasing, decreasing, or constant.



Example 2: Find intervals where $a)\ f(x) = x^2,\ \ b)\ f(x) = x^3$ and $c)\ f(x) = 1/x$ is:

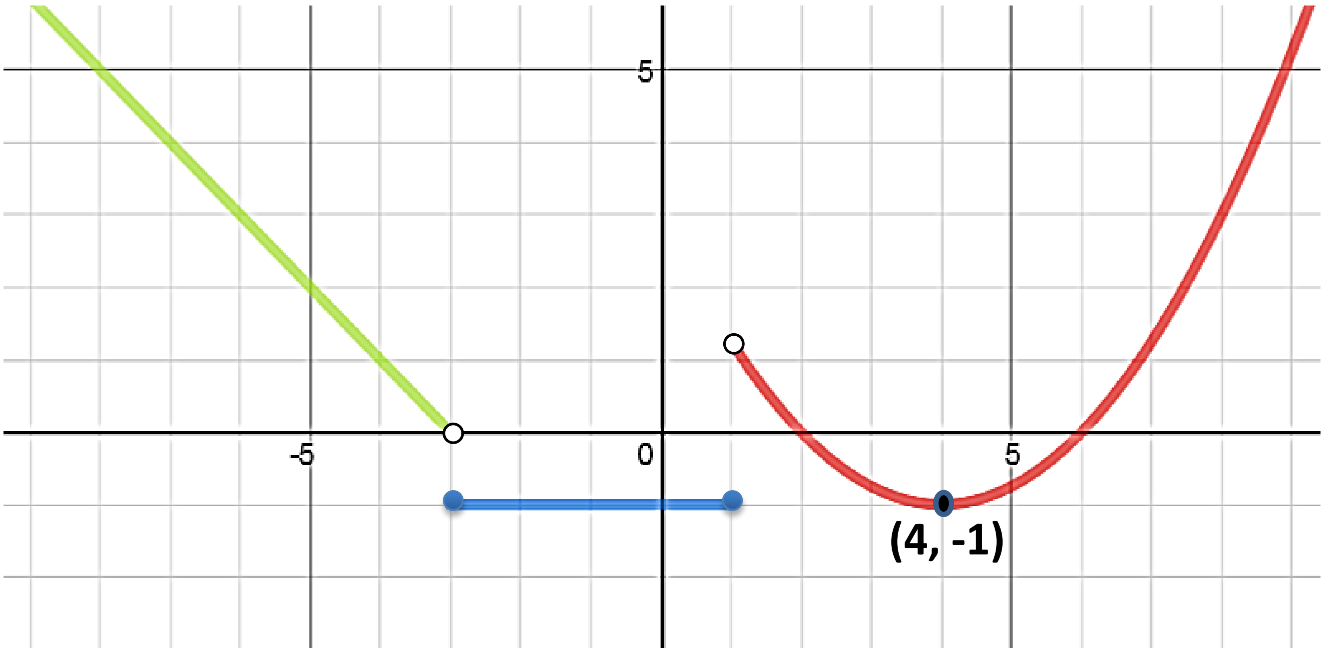
a) Increasing

b) Decreasing

OER West Texas A & M Tutorial 32: Graphs of Functions, Part II

Example 3: Find the interval where the function f is **increasing**, **decreasing** or a **constant**

$$f(x) = \begin{cases} \left(\frac{1}{2}x - 2\right)^2 - 1 & if\ x > 1 \\ y = -1 & if\ -3 \le x \le 1 \\ -x - 3 & if\ x < -3 \end{cases}$$



Even and Odd Functions and Symmetry:

Definition (even function)

A function f is even if $f(-x) = f(x)$ for all x in the domain of f

- An even function has graph that is **symmetric** with respect to (**wrt**) the **y-axis**

Definition (odd function)

A function f is odd if $f(-x) = -f(x)$ for all x in the domain of f

- An odd function has graph that is **symmetric** with respect to (**wrt**) the **origin**.

Example 1: Determine whether each of the functions is even, odd, or neither.

a) $f(x) = -3x^3 + 2x$

Solution: $f(-x) = -3(-x)^3 + 2(-x) = -3(-x^3) - 2x = 3x^3 - 2x = -(-3x^3 + 2x) = -f(x)$

Thus, the function is odd.

b) $f(x) = 3x^2 - 2x + 5$

Solution:

$$\begin{aligned} f(-x) &= 3(-x)^2 - 2(-x) + 5 \\ &= 3(x^2) + 2x + 5 \\ &= 3x^2 + 2x + 5 \end{aligned}$$

$$f(-x) \neq -(3x^2 - 2x + 5) = -3x^2 + 2x - 5 = -f(x) \ or$$

$$f(-x) \neq f(x)$$

Thus, the function is **neither even nor odd**

c) $f(x) = 5x^4 + 2x^2 - 1$ (is even, show)

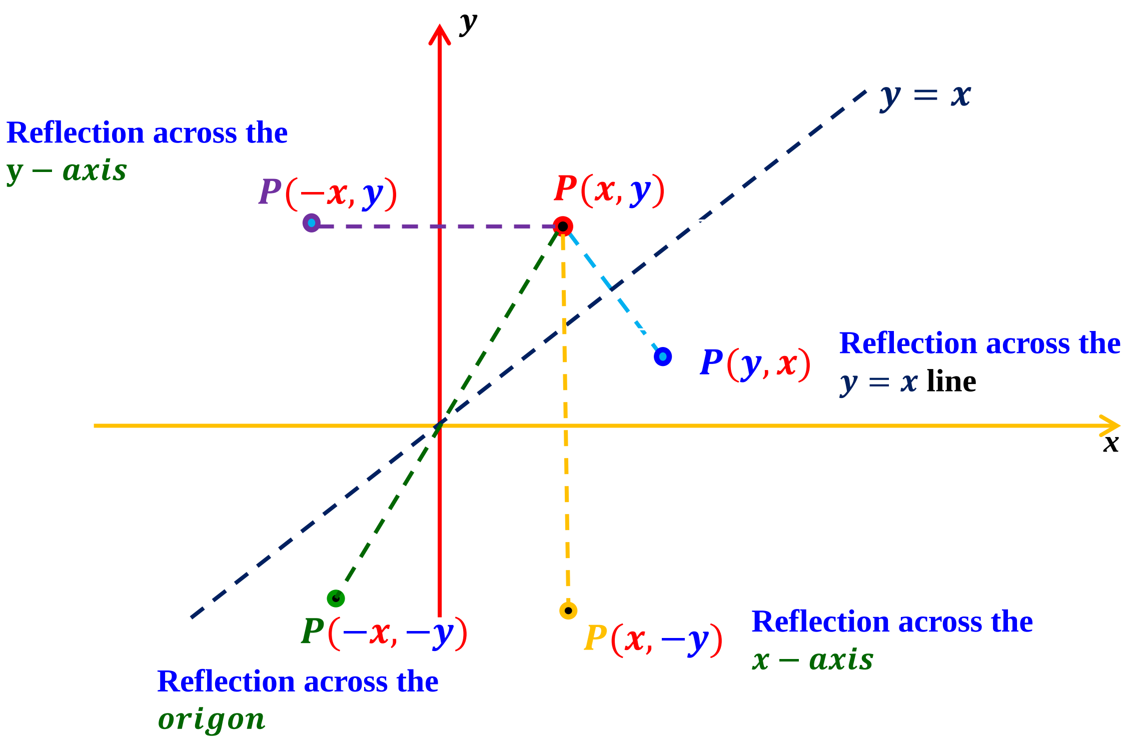
Example 2: Describe the following functions as **even, odd** or **neither** and **justify**

a) $f(x) = x^4 - x^2 + 12$ b) $f(x) = x^3 + 2x$ c) $f(x) = x^4 + x^3$

Symmetry

Given a point $P(x, y)$

- $(x, -y)$ is a point of **symmetry** of the **point P** with respect to the **x-axis**
- $(-x, y)$ is a point of **symmetry** of the **point P** with respect to the **y-axis**
- $(-x, -y)$ is a point of **symmetry** of the **point P** with respect to the **origin**



OER West Texas A & M Tutorial 32: **Graphs of Functions, Part II**

Homework: Exercise 1.6.2: page 107 #21 – 41 (Stitz and Zeager Book)

The Algebra of Functions (Page 76)

Basic Operations between two functions:

If f and g are functions and x is in the domain of each function, then we define the **sum**, **difference**, **product**, and **quotient** of f and g as follows

	Definition	**Domain**
Sum:	$(f+g)(x)=f(x)+g(x)$	$Domain\ of\ f \cap Domain\ of\ g$
Difference:	$(f-g)(x)=f(x)-g(x)$	$Domain\ of\ f \cap Domain\ of\ g$
Product:	$(f\times g)(x)=f(x)\times g(x)$	$Domain\ of\ f \cap Domain\ of\ g$
Quotient:	$\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}$	$Domain\ of\ f \cap Domain\ of\ g$ ***excluding*** $\{x \mid g(x)\neq 0\}$

Example 1: Given $f(x)=x^2-3$ and $g(x)=2x+1$, find the following:

$a)\ (f+g)(x)=f(x)+g(x)$
$$=(x^2-3)+(2x+1)$$
$$=x^2+2x-2$$

$b)\ (f\,g)(x)=f(x)g(x)$
$$=(x^2-3)(2x+1)$$
$$=2x^3+x^2-6x-3$$

c) $(f-g)(x)$

$d)\ (f+g)(5)=f(5)+g(5)$
$$=(5^2-3)+(2(5)+1)$$
$$=(25-3)+(10+1)$$
$$=22+11=33$$

e) $(f\,g)(2)$ f) $\left(\frac{f}{g}\right)(x)$ g) $\left(\frac{f}{g}\right)\left(-\frac{1}{2}\right)$

Example 2: Find the domain of f, g, $f+g$, $f-g$, $f\times g$, $\frac{f}{g}$, $\frac{g}{f}$ where

$$f(x)=x^2-3 \text{ and } g(x)=2x+1$$

Domain of f: $(-\infty,\infty)$

Domain of g: $(-\infty,\infty)$

Domain of $f+g$: $(-\infty,\infty)$

Domain of $f\,g$: $(-\infty,\infty)$

Domain of $\frac{f}{g}$: $\left(-\infty,-\frac{1}{2}\right)\cup\left(-\frac{1}{2},\infty\right)$

Domain of $\frac{g}{f}$: $(-\infty,-\sqrt{3})\cup(-\sqrt{3},\ \sqrt{3})\cup(\sqrt{3},\infty)$

Example 3: Let f be the function defined by

$$f=\{(-3,4),(-2,2),(-1,0),(0,1),(1,3),(2,4),(3,-1)\}$$

and let g be the function defined by

$$g=\{(-3,-2),(-2,0),(-1,-4),(0,0),(1,-3),(2,1),(3,2)\}$$

Compute the indicated value if it exists.

a) $(f+g)(-3)$ c) $(f-g)(2)$ e) $(f\times g)(-1)$ g) $(g\circ f)(-1)$

b) $(f\div g)(-2)$ d) $(g\div f)(-3)$ f) $(f\circ g)(-3)$

Example 4: a) Find the domain of $f,\ g,\ f+g,\ f-g,\ fg,\ \frac{f}{g},\ \frac{g}{f}$ where

$$f(x) = x + 2 \ \ and \ \ g(x) = \sqrt{x-1}$$

b) Find $(f+g)(x)$, $(f-g)(x)$, $(f \times g)(x)$, $(f \div g)(x)$, and $(g \div f)(x)$.

Composite Functions (page 359)

Definition: The **composite function** $f \circ g$, the composition of f *and* g**,** is defined as

$(f \circ g)(x) = f(g(x))$, where x is in the domain of g and $g(x)$ is in the domain of f**.**

Example 5: Example 3: Given $f(x) = x^2 - 3$ and $g(x) = 2x + 1$, find

a) $(f \circ g)(x)$
b) $(f \circ g)(1)$
c) $(g \circ f)(-x)$
d) $(g \circ f)(-2)$
e) $(f \circ f \circ f)(1)$

Solution: a)

$$(f \circ g)(x) = f(g(x)) = f(2x+1) = (2x+1)^2 - 3 = 4x^2 + 4x - 2$$

Decomposition of Functions

In decomposing a function we will make **two** functions out of the **given function**.

Example 6: Find $f(x)$ and $g(x)$ such that $h(x) = (f \circ g)(x)$.

a) Decompose the function $h(x) = (4 + 3x)^5$

Solution: a) We can write $h(x)$ as:

$h(x) = (f \circ g)(x) = f(g(x))$, where $g(x) = 4 + 3x$ and $f(x) = x^5$

b) Decompose the function $p(x) = \sqrt{x^2 + 4}$

c) Decompose the function $r(x) = e^{5x-3}$

d) Decompose the function $f(x) = \frac{3}{x^2 - 2x}$

Examples 5.1.1, 5.1.2, & 5.1.3: Homework, Reading Page 360 – 367

OER West Texas A&M Tutorial 30B: Operations with Functions

Homework: Exercise 1.5.1 Page 84 #1 – 20 & #51 – 62

Exercise 5.1.1 page 369 # 1 – 40 & 44 – 55 (Stitz and Zeager Book)

Transformations of Functions

Transformations:

Translations and **Reflections; Vertical and Horizontal Shrinks and Stretches**

Translations

1) Vertical Translation: $y = f(x) \pm c, for\ c > 0$

The graph of $y = f(x) + c$ is the graph of $y = f(x)$ shifted vertically c units up

The graph of $y = f(x) - c$ is the graph of $y = f(x)$ shifted vertically c units down

2) Horizontal Translations: $y = f(x \pm c), for\ c > 0$

The graph of $y = f(x - c)$ is the graph of $y = f(x)$ shifted horizontally c units to the right

The graph of $y = f(x + c)$ is the graph of $y = f(x)$ shifted horizontally c units to the left.

Reflections

1) Across the x-axis:

The graph of $y = -f(x)$ is the reflection of the graph of $y = f(x)$ across the **x-axis**.

2) Across the y-axis:

The graph of $y = f(-x)$ is the **reflection** of the graph of $y = f(x)$ across the **y-axis**.

Graphs; Stretches and Shrinks

1) Vertical Stretching and shrinking

To graph $y = cf(x)$:

- If $c > 1$, **stretch** the graph of $y = f(x)$ **vertically** by a **factor of c**
- If $0 < c < 1$, **shrink** the graph of $y = f(x)$ **vertically** by a **factor of c**

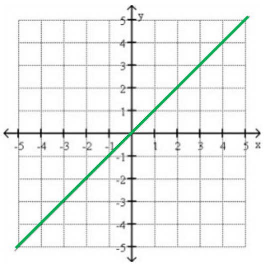
2) Horizontal Stretching and shrinking

To graph $y = f(cx)$:

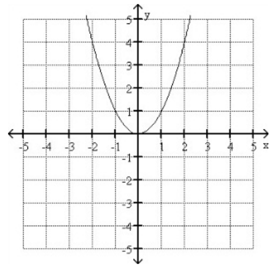
- If $c > 1$, **shrink** the graph of $y = f(x)$ **horizontally** by a **factor of $1/c$**
- If $0 < c < 1$, **stretch** the graph of $y = f(x)$ **horizontally** by a **factor of $1/c$**

Recall the basic graphs

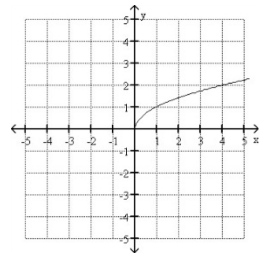
$f(x) = x$



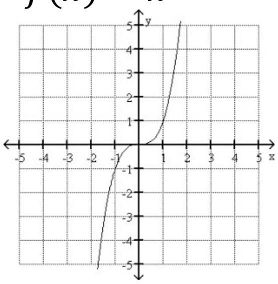
$f(x) = x^2$



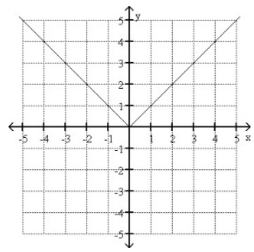
$f(x) = \sqrt{x}$



$f(x) = x^3$



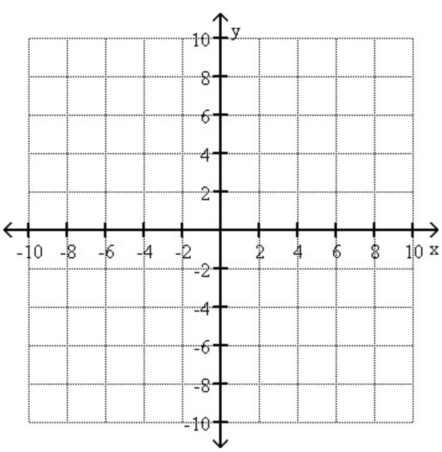
$f(x) = |\,x\,|$



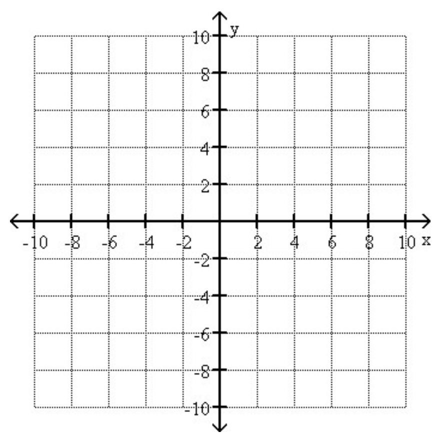
Example 1: Using the given information sketch the graph and give the equation.

Given $\boldsymbol{f(x) = x^2}$

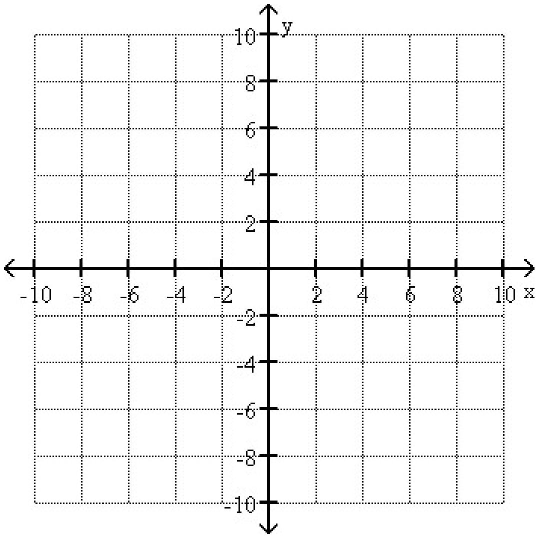
a) $\boldsymbol{f(x) = x^2 + 3}$



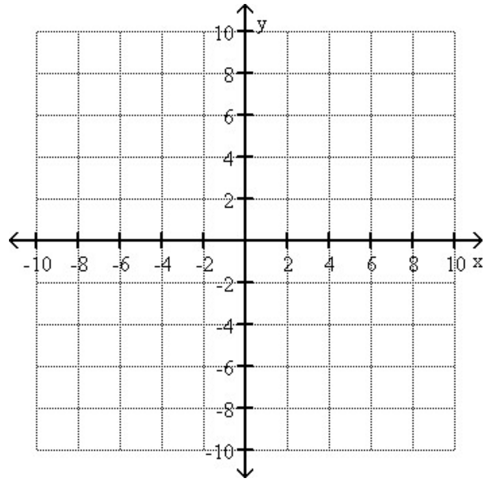
b) $\boldsymbol{f(x) = -(x+1)^2 + 2}$



c) $\boldsymbol{f(x) = (x-1)^2 + 1}$



d) The graph of $\boldsymbol{f(x) = x^2}$, but **upside down**, and **shifted left 2 units**



Example 2: Given $\boldsymbol{y = \sqrt{x}}$ sketch the graph or give the equation

a) The graph of $\boldsymbol{f(x) = \sqrt{x}}$, but shifted left 4 units

b) The graph of $\boldsymbol{y = \sqrt{x-2}}$

c) The graph of $\boldsymbol{y = -\sqrt{x} + 1}$

Example 3: Using **reflection**, **horizontal** and **vertical shifts** and the graph of $\boldsymbol{y = |x|}$ sketch

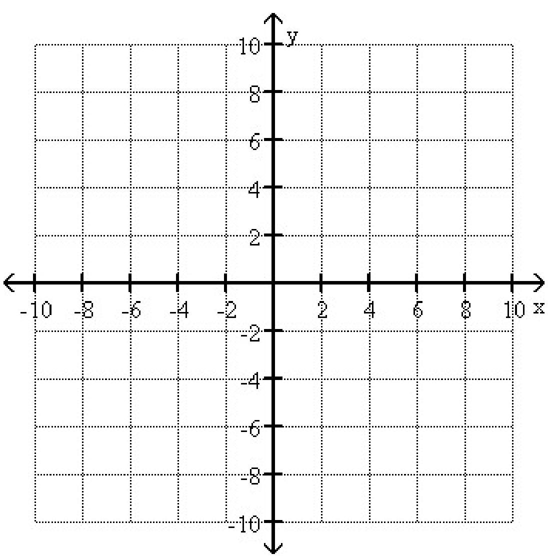
a) $f(x) = -|x|$

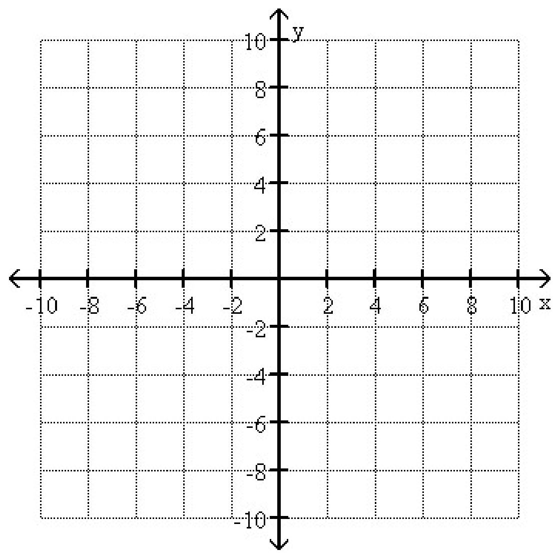
b) $f(x) = |x - 1| + 1$

c) $f(x) = |x - 2| - 1$

d) $f(x) = |x + 1| + 1$

e) $f(x) = -|x + 2| + 3$





One – to – One Functions and Inverse Functions (Page 378)

Objectives: By the end of this section you should be able to

- Identify one – to – one functions
- Find the inverse of a function and domain and range of inverse functions
- Use the Horizontal line test
- Identify one – to – one functions from graphs
- Graph inverse functions
- State the inverse function property

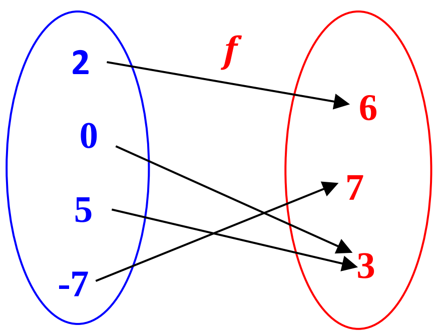
One – to – one Function

Definition:

- A function f is said to be a **one – to – one function if and only if** for every a, b in the domain of f $a \neq b$ **implies** that $f(a) \neq f(b)$. That is f is **one – to –one if and only if different inputs** always have **different outputs. Or equivalently**
- A function f is said to be a **one – to – one function** if every **y** in the **range** related to **exactly one** x in the **domain. Or equivalently**
- A function f is said to be a **one – to – one** if every **horizontal** line **intersects** the graph of f at **most once**. (**Horizontal Line Test**)

1. Venn Diagrams (Example 1)

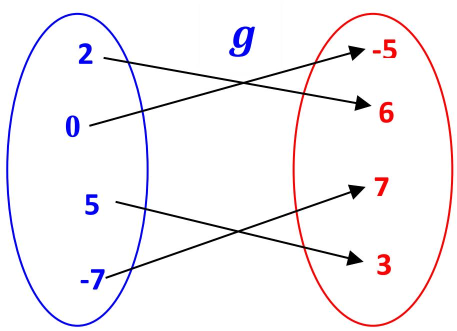
a)



$f = \{(2, 6), (0, 3), (5, 3), (-7, 7)\}$

f **is not a one – to – one function. Why?**

b)

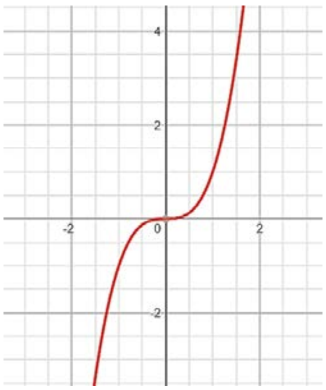


$g = \{(2, 6), (0, -5), (5, 3), (-7, 7)\}$

g is a **one – to – one function. Why?**

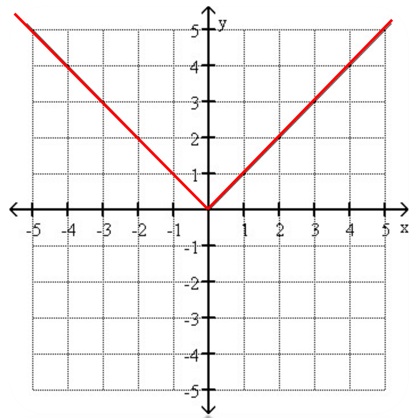
2. Graphs (Example 2)

d) $f(x) = x^3$



One – to – one, why?

e) $f(x) = |\,x\,|$



Not one – to – one, why?

Example 3: Verify the following functions are **one – to – one**.

a) $g = \{(2,6),(0,-5),(5,3),(-7,7)\}$

b) $f(x) = 3x - 2$

c) $f(x) = \frac{1}{x}$

d) $g(x) = \sqrt{x}\,,\ x \geq 0$

e) $h(x) = x^3$

f) $f(x) = (x-1)^2 - 2,\ x \geq 1$

Example 5.2.1 page 382 reading

The Inverse of a Function

Definition: **The inverse** of a function f is a relation defined as the set $\{(y,\ x)\colon whenever\ (x,y)\ belongs\ to\ f\ \}$

Example 4: Find the inverse

a) $f = \{(2,\ 6),(0,3),(5,3),(-7,7)\}$

Solution: $Inverse\ of\ f = \{(6,\ 2),(3,\ 0),(3,\ 5),(7,-7)\}$

Note: The inverse of f is **not** a **function**. **Why?**

b) $g = \{(2,6),(0,-5),(5,3),(-7,7)\}$

Solution: Inverse of $g = \{(6,\ 2),(-5,\ 0),(3,\ 5),(7,-7)\}$

Here: The Inverse of g is a function**. Why?**

Note: The inverse of a function may not always be a function

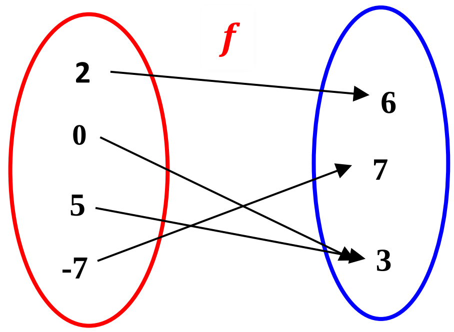
Question:

- When is the **inverse** of a function, a **function?**
- In other words, which function inverse gives **inverse function**?

To answer these questions consider the following Venn Diagrams

Example 5: Venn Diagrams

a)

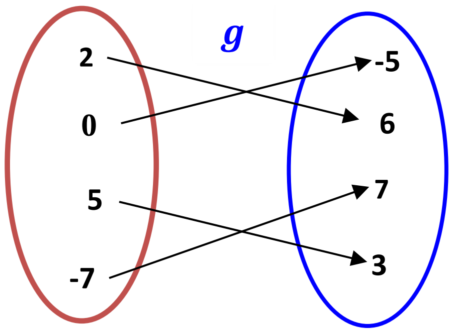


$\boldsymbol{f} = \{(\mathbf{2},\ \mathbf{6}), (\mathbf{0},\mathbf{3}), (\mathbf{5},\mathbf{3}), (-\mathbf{7},\mathbf{7})\ \}$

Inverse of f =

Is the inverse of f a function?

b)



$\boldsymbol{g} = \{(\mathbf{2},\mathbf{6}), (\mathbf{0},-\mathbf{5}), (\mathbf{5},\mathbf{3}), (-\mathbf{7},\mathbf{7})\ \}$

Inverse of g =

Is the inverse of g a function?

Note: The inverse of a function f is a function if the original function is **one-to-one**

Inverse Functions

Definition: The inverse of a **one-to-one function** f , written as f^{-1}**,** is a function given by:

$$f^{-1} = \{(y,\ x): \text{whenever } (x, y) \text{ belongs to } f\ \}.$$

Note:

- If f is a **one – to – one function,** we call f^{-1} the **inverse function of** f
- If f takes x in to y the inverse function f^{-1} takes y in to x**,** that is; if $y = f(x)$, then $x = f^{-1}(y)$

Inverse function Property

Let f be a **one – to – one function** with **domain A** and **range B**. The inverse function f^{-1} has **domain B** and **range A** and **satisfies** the following **cancellation properties**:

$f^{-1}(f(x)) = x$ for every x in **A**

$f\left(f^{-1}(x)\right) = x$ for every x in **B**

Conversely any **function** f^{-1} **satisfying** these cancellation equations is the **inverse** of f

Finding the Inverse Functions from Equations

Procedures for finding the inverse function f^{-1}:

1) Write y for $f(x)$
2) Solve for x in 1)
3) 2) gives $x = f^{-1}(y)$
4) Finally, in 3) replace **y with** x; that is, write the equation in terms of x

Example 5.2.2 page 384 reading

Example 6: For each of the following functions find the inverse function

a) $g = \{(2,6),(0,-5),(5,3),(-7,7)\}$

b) $f(x) = 3x - 2$

c) $f(x) = \frac{1}{x}$

d) $g(x) = \sqrt{x},\ x \geq 0$

e) $h(x) = x^3$

f) $f(x) = (x-1)^2 - 2,\ x \geq 1$

Solutions:

b) $f(x) = 3x - 5$, replace $f(x)$ with **y**

$y = 3x - 5$, solve for x

$x = \frac{1}{3}y - \frac{5}{3}$, **thus the inverse function is** $f^{-1}(y) = \frac{1}{3}y - \frac{5}{3}$

Finally replacing **y with** x we get

$f^{-1}(x) = \frac{1}{3}x - \frac{5}{3}$

d) $g(x) = \sqrt{x},\ x \geq 0$, **replace** $g(x)$ **by y**

$y = \sqrt{x}$, Solve for **y.** Note $x \geq 0$ implies $y \geq 0$

$x = y^2$, Squaring both sides

Thus, $f^{-1}(y) = y^2,\ y \geq 0$, replacing **y** ***with*** x we get

$f^{-1}(x) = x^2,\ x \geq 0$, the inverse function

f) $f(x) = (x-1)^2 - 2,\ x \geq 1$, **replace** $f(x)$ with **y**

$y = (x-1)^2 - 2,\ x \geq 1$, **solve for** x

$(x-1)^2 = y + 2$, which implies

$x - 1 = \pm\sqrt{y+2}$, since $x \geq 1$, **i.e.** $x - 1 \geq 0$ we take the positive square root

$x - 1 = \sqrt{y+2}$, which gives

$x = 1 + \sqrt{y+2}$

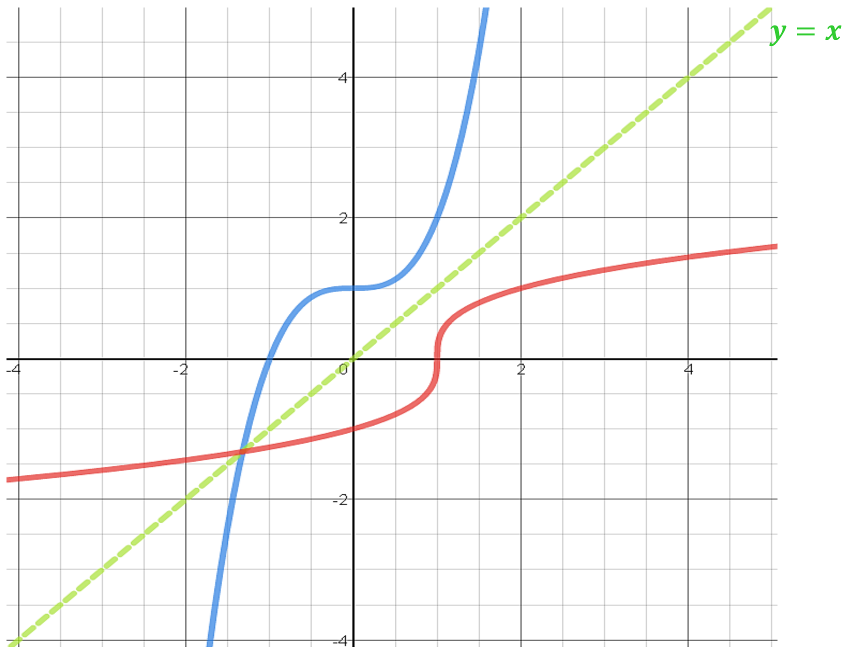
Thus, $f^{-1}(x) = 1 + \sqrt{x+2},\ x \geq -2$, inverse function of f

Graphs of Inverse Functions

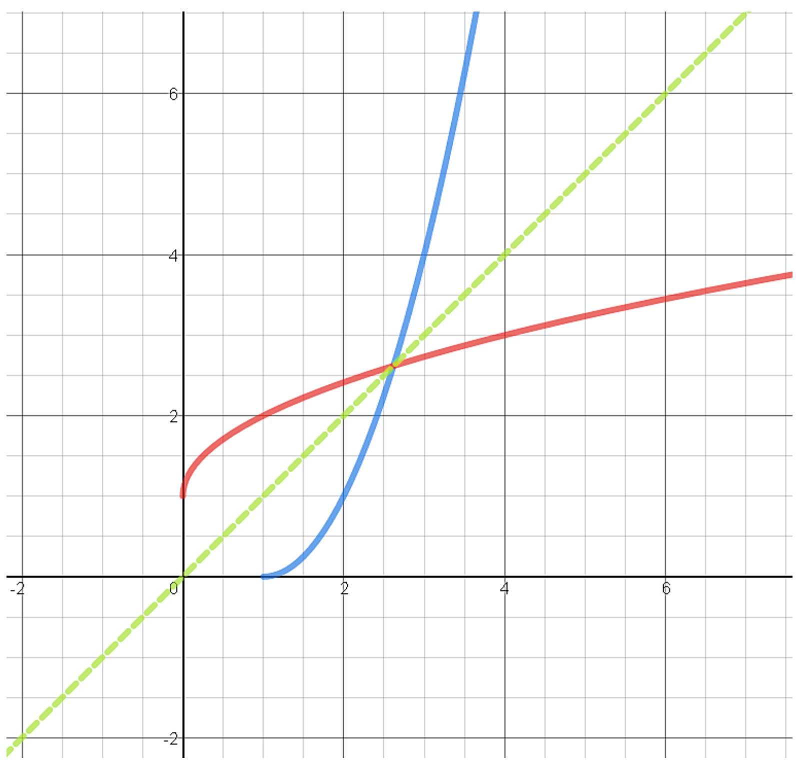
Recall: The reflection of a point $(\boldsymbol{a},\ \boldsymbol{b})$ about the line $\boldsymbol{y = x}$ is the point $(\boldsymbol{b},\ \boldsymbol{a})$, but $(\boldsymbol{b},\ \boldsymbol{a})$ is a point in $\boldsymbol{f^{-1}}$ whenever $(\boldsymbol{a},\ \boldsymbol{b})$ is in $\boldsymbol{f}$. Thus**,** the graph of $\boldsymbol{f^{-1}}$ is the **reflection** about the line $\boldsymbol{y = x}$ of the graph of $\boldsymbol{f}$.

Example 5.2.3 page 389 reading

Example 1: Graph of $\boldsymbol{f(x) = x^3 + 1}$ and its inverse $\boldsymbol{f^{-1}(x) = \sqrt[3]{x - 1}}$. Graph of $\boldsymbol{f^{-1}}$is the reflection of the graph of $\boldsymbol{f}$ across the line $\boldsymbol{y = x}$



Example 2: Graph of $\boldsymbol{f(x) = (x-1)^2, x \geq 1}$ and its inverse $\boldsymbol{f^{-1}(x) = \sqrt{x} + 1}$



Example 3: For each of the following functions **sketch** the **inverse function**

g) $g = \{(2,6),(0,-5),(5,3),(-7,7)\}$

h) $f(x) = 3x - 2$

i) $f(x) = \frac{1}{x}$

j) $g(x) = \sqrt{x},\ x \geq 0$

k) $h(x) = x^3$

l) $f(x) = (x-1)^2 - 2,\ x \geq 1$

OER West Texas A&M Tutorial 32B: Inverse Functions

Homework

Exercise 5.2.1 page 394 # 1 – 24 (Stitz and Zeager Book)

Practice Test

OER West Texas A&M Tutorial 33: Practice Test on Tutorials 25 - 32