2.5 Infinite Limits

Definition of Infinite Limits

Let $f(x)$ be defined for all x in some open interval containing c, except that $f(x)$ need not be defined at c. We will write $\lim\limits_{x \to c} f(x) = \infty$ if given any positive number M, there exists a number $\delta > 0$ such that $f(x) > M$ if $0 < |x - c| < \delta$.

Similarly, we will write $\lim\limits_{x \to c} f(x) = -\infty$
if given any negative number N, there exists a number $\delta > 0$ such that $f(x) < N$ if $0 < |x - c| < \delta$.

We can define $\lim\limits_{x \to c^+} f(x) = \infty$, $\lim\limits_{x \to c^-} f(x) = \infty$, $\lim\limits_{x \to c^+} f(x) = -\infty$, and $\lim\limits_{x \to c^-} f(x) = -\infty$
by making appropriate modifications to the inequality $0 < |x - c| < \delta$.

Example 1 Find each limit.

a. $\lim\limits_{x \to 3^+} \dfrac{1}{x - 3}$ b. $\lim\limits_{x \to 3^-} \dfrac{1}{x - 3}$ c. $\lim\limits_{x \to 3} \dfrac{1}{x - 3}$ d. $\lim\limits_{x \to 3} \dfrac{1}{(x - 3)^2}$

Example 2 Find each limit.

a. $\lim\limits_{x \to 3} \dfrac{2x - 2}{x^2 - 5x + 4}$ b. $\lim\limits_{x \to 1} \dfrac{2x - 2}{x^2 - 5x + 4}$ c. $\lim\limits_{x \to 4} \dfrac{2x - 2}{x^2 - 5x + 4}$

Vertical Asymptotes

Let f and g be continuous on an open interval containing c. If $f(c) \neq 0$, $g(c) = 0$, and there exists an open interval containing c such that $g(x) \neq 0$ for all $x \neq c$ in the interval, then the graph of the function given by

$$h(x) = \frac{f(x)}{g(x)}$$

has a vertical asymptote at $x = c$.

Example 3 Find all vertical asymptotes of each function.

a. $f(x) = \dfrac{x-3}{x^2-9}$

b. $g(x) = \dfrac{x^3-6x^2-x+30}{x^2+6x+8}$

Properties of Infinite Limits

Let c and L be real numbers and let f and g be functions such that $\lim\limits_{x\to c} f(x) = \infty$ and $\lim\limits_{x\to c} g(x) = L$.

1. $\lim\limits_{x\to c}[f(x) \pm g(x)] = \infty$
2. $\lim\limits_{x\to c}[f(x)g(x)] = \infty$ if $L > 0$ and $\lim\limits_{x\to c}[f(x)g(x)] = -\infty$ if $L < 0$.
3. $\lim\limits_{x\to c} \dfrac{g(x)}{f(x)} = 0$

Example 4 Find each limit.

a. $\lim\limits_{x\to \frac{\pi}{2}^-} x \sec x$

b. $\lim\limits_{x\to \frac{\pi}{2}^+} x \sec x$

c. $\lim\limits_{x\to 0^+} (\sin x + \csc x)$

Our textbook does not place much emphasis on using the M - δ definition to show that a particular limit is ∞ or $-\infty$, so neither will I. If this sort of thing excites you, the following example may interest you. On the other hand, if its effect on you is similar to that of drinking Ipecac syrup, you might want to look the other way.

Example 5 Prove that $\lim_{x\to 4} \frac{1}{(x-4)^2} = +\infty$.

Let $M > 0$ be an arbitrary positive number. We must first find a positive number δ (basically, this is an algebraic expression with M in it) such that $\frac{1}{(x-4)^2} > M$ whenever $0 < |x-4| < \delta$. Then we must show through a sequence of logical statements that the expression for δ actually satisfies this property.

Step 1 Find an expression for δ.

Let's begin with the inequality $\frac{1}{(x-4)^2} > M$ and see if we can express $|x-4|$ in terms of M.

Taking the reciprocal of both sides, we obtain $(x-4)^2 < \frac{1}{M}$.

Since both sides of the inequality are positive, and the square root function is an increasing function, taking square roots of both sides preserves the direction of the inequality.

Since $\sqrt{(x-4)^2} = |x-4|$, we now have $|x-4| < \sqrt{\frac{1}{M}}$.

So, it looks like $\sqrt{\frac{1}{M}}$ is a good candidate for our δ.

Step 2 Show that $\delta = \sqrt{\frac{1}{M}}$ *actually works.*

Let $M > 0$ be an arbitrary positive number. Let $\delta = \sqrt{\frac{1}{M}}$ and suppose $0 < |x-4| < \delta$.

By taking reciprocals, the inequality $0 < |x-4| < \delta$ gives us the equivalent inequality $0 < \frac{1}{\delta} < \frac{1}{|x-4|}$, so $\frac{1}{|x-4|} > \frac{1}{\delta}$. Consequently, $\frac{1}{|x-4|^2} > \frac{1}{\delta^2}$.

Then $\frac{1}{(x-4)^2} = \frac{1}{|x-4|^2} > \frac{1}{\delta^2} = \frac{1}{\left(\sqrt{\frac{1}{M}}\right)^2} = \frac{1}{\frac{1}{M}} = M$.

This shows that $\frac{1}{(x-4)^2} > M$ whenever $0 < |x-4| < \delta$ and proves that $\lim_{x\to 4} \frac{1}{(x-4)^2} = +\infty$.