5.5 Integration by Substitution

In this section we use the method of substitution to take expressions which we *do not know* how to integrate and write them as simpler expressions which we *do know* how to integrate.

Remember the chain rule? $\frac{d}{dx}[F(g(x))] = F'(g(x)) \cdot g'(x)$
If we rewrite this equation using integral notation, it looks something like this:

$$\int F'(g(x)) \cdot g'(x)dx = F(g(x)) + C$$

The integrand looks kind of messy, but it can be easily integrated because of
the special form that it is in.

In this section we will consider integrals that can be expressed in the form $\int f(g(x)) \cdot g'(x)dx$.
Notice that this integrand has a little f rather than a big F'.

The function f will usually be a familiar function whose antiderivative F we can readily find.
If F is any antiderivative of f, we will be able to write $\int f(g(x)) \cdot g'(x)dx = F(g(x)) + C$.

Example 1 For each expression, find $f(x)$ and $g(x)$ so that the expression is in the form $kf(g(x)) \cdot g'(x)$ for some constant k.

a. $\sin(3x^2 - 4) \cdot (6x)$
Let $g(x) = 3x^2 - 4$.
Then $g'(x) = 6x$, so if
we let $f(x) = \sin x$, then
$f(g(x)) \cdot g'(x) = \sin(3x^2 - 4) \cdot (6x)$
The constant k is equal to 1.

b. $(x^2 - 5x)^5(4x - 10)$

c. $\sin^3(2x)\cos(2x)$

d. $x^3\sqrt{x^4 - 16}$

In practice, we will make a change of variables to perform the integration.

If we are trying to integrate $\int f(g(x)) \cdot g'(x)dx$, we usually will let $u = g(x)$.

Then $\frac{du}{dx} = g'(x)$, so that in differential notation we have $du = g'(x)dx$.

Substituting this into the original integrand gives us $\int f(u)du$.

If F is any antiderivative of f, we can integrate $\int f(u)du$ as follows:

$\int f(u)du = F(u) + C$

Since the original integrand was expressed in terms of x, we will want to leave our final answer that way.

$$\int f(g(x)) \cdot g'(x)dx = \int f(u)du = F(u) + C = F(g(x)) + C$$

Example 2 Rewrite each integral. First, factor out an appropriate constant if necessary to write the integral in the form $k\int f(g(x)) \cdot g'(x)dx$. Let $u = g(x)$, find $\frac{du}{dx} = g'(x)$, and express it in the form $du = g'(x)dx$ so that $k\int f(g(x)) \cdot g'(x)dx$ is rewritten as $k\int f(u)du$.

a. $\int \sin(3x^2 - 4)(6x)dx$

Let $u = 3x^2 - 4$.

Then $\frac{du}{dx} = 6x$.

and $du = 6x dx$.

So $\int \sin(3x^2 - 4)(6x)dx = \int \sin(u)du$.

b. $\int (x^2 - 5x)^5(4x - 10)dx$

c. $\int \sin^3(2x)\cos(2x)dx$

d. $\int x^3\sqrt{x^4 - 16}\,dx$

Integration by Substitution

Step 1. Make a choice for u, say $u = g(x)$.

Step 2. compute $\frac{du}{dx} = g'(x)$.

Step 3. Make the substitution $u = g(x),\ du = g'(x)dx$.

At this stage, the entire integral must be in terms of u; no x's should remain.

If this is not the case, try a different choice of u.

Step 4. Evaluate the resulting integral, if possible.

Step 5. Replace u by $g(x)$, so that the final answer is in terms of x.

Example 3 Evaluate each integral.

a. $\int (5x^2 - 3x + 2)^9 (10x - 3)dx$

b. $\int x^3 \cos(6x^4)dx$

c. $\int \cos x \sqrt{\sin x}\, dx$

d. $\int \frac{3x}{\sqrt{x - 2}} dx$

Evaluating Definite Integrals by Substitution

Theorem If g' is continuous on $[a, b]$ and f is continuous and has an antiderivative on an interval containing the values of $g(x)$ for $a \le x \le b$, and $u = g(x)$, then
$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.$

Example 4 Evaluate $\int_{\pi/6}^{3\pi/2} 6\sin^2\theta\cos\theta d\theta.$

Make the substitution $u = \sin\theta,$ and replace the limits by $u = \sin\left(\frac{\pi}{6}\right)$ and $u = \sin\left(\frac{3\pi}{2}\right)$.

Then evaluate the resulting definite integral in the variable u.

Example 5 Evaluate each definite integral.

a. $\int_1^2 \frac{72x dx}{(4x^2 - 1)^3}$

b. $\int_7^{12} \frac{x^2 dx}{\sqrt{x - 3}}$