3.7 Related Rates

Example 1

Imagine a rectangle with adjacent sidelengths denoted by x and y where the quantities x and y both vary with time t. At some times the rectangle may be tall and skinny, and at other times the rectangle may be short and fat. If x and y are differentiable functions of t, the quantities $\frac{dx}{dt}$ and $\frac{dy}{dt}$ denote the rates at which the respective sidelengths are changing with respect to time. Let A denote the area of our imaginary rectangle. Since $A = xy$, the area A is a function of x and y and, consequently, A is also a function of t. It is reasonable to ask, then, how the area A is changing with respect to time at any particular time t. That is, we seek $\frac{dA}{dt}$.

To find $\frac{dA}{dt}$, we will differentiate each side of the equation $A = xy$ with respect to t.

$$\frac{d}{dt}[A] = \frac{d}{dt}[xy]$$
$$\frac{dA}{dt} = x\frac{d}{dt}[y] + y\frac{d}{dt}[x]$$ <——— Here, we are using the product rule.
$$\frac{dA}{dt} = x\frac{dy}{dt} + y\frac{dx}{dt}$$

Now, suppose that at a particular instant in time, the following quantities are known:
$x = 3.0$ m, $y = 2.0$ m, $\frac{dx}{dt} = 0.2$ m/s, and $\frac{dy}{dt} = 0.1$ m/s.
Then $\frac{dA}{dt} = (3.0 \text{ m})(0.1 \text{ m/s}) + (2.0 \text{ m})(0.2 \text{ m/s}) = 0.7 \text{ m}^2/\text{s}$.

To lend a bit more meaning to the quantity 0.7 m^2/s, suppose the quantities $\frac{dx}{dt}$ and $\frac{dy}{dt}$ were to remain constant for a full second. During that second, x would increase from 3.0 m to 3.2 m and y would increase from 2.0 m to 2.1 m. The area A would then increase from 6.0 m^2 to 6.72 m^2. So, hopefully, it seems reasonable to say that the area A was increasing at the instantaneous rate of 0.7 m^2/s.

Guidelines for solving related rates problems:

Step 1. Draw a figure and label the quantities that vary.

Step 2. Identify the rates of change that are known and the rate of change that is to be found.

Step 3. Find an equation that relates the quantity whose rate of change is to be found to the quantities whose rates of change are known.

Step 4. Differentiate both sides of this equation with respect to time and solve for the derivative that will give the unknown rate of change.

Step 5. Evaluate this derivative at the appropriate point.

Example 2 Let L be the length of a diagonal of a rectangle whose sides have lengths x and y, and assume that x and y vary with time. If x increases at a constant rate of 0.5 ft/s and y decreases at a constant rate of 0.25 ft/s, how fast is the size of the diagonal changing when $x = 5$ ft and $y = 12$ ft?

$$L^2 = x^2 + y^2$$

$$\frac{d}{dt}[L^2] = \frac{d}{dt}[x^2] + \frac{d}{dt}[y^2]$$

$$2L\frac{dL}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\frac{dL}{dt} = \frac{2x\frac{dx}{dt} + 2y\frac{dy}{dt}}{2L}$$

$$\frac{dL}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{L}$$

$$\frac{dL}{dt} = \frac{(5\text{ ft})(0.5\text{ ft/s}) + (12\text{ ft})(-0.25\text{ ft/s})}{13\text{ ft}}$$

$$\frac{dL}{dt} = \frac{(2.5\text{ ft}^2\text{/s}) + (-3\text{ ft}^2\text{/s})}{13\text{ ft}}$$

$$\frac{dL}{dt} = \frac{-0.5\text{ ft}^2\text{/s}}{13\text{ ft}} \approx -0.03846\text{ ft/s}$$

Example 3 An aircraft is flying horizontally at a constant height of 4000 ft above the ground moving away from a fixed observation point on the ground. At a certain instant the angle of elevation θ to the aircraft is 30^o and the speed of the aircraft is 440 ft/s.

(a) How is θ changing at this instant? Express the result in units of degrees/s.

(b) How fast is the distance between the aircraft and the observation point changing at this instant? Express the result in units of ft/s.

Example 4 A conical tank with vertex down has a radius of 4 ft at the top and is 10 ft high. If a fluid flows into the tank at a rate of 8 cubic feet per minute, how fast is the depth of the fluid increasing when the fluid is 6 ft deep?