5.2 Area

THE AREA PROBLEM. Given a function f that is continuous and nonnegative on an interval $[a,b]$, find the area between the graph of f and the interval $[a,b\,]$on the x-axis.

The Rectangle Method for Finding Areas

(1) Divide the interval $[a,b]$ into n equal subintervals, and over each subinterval construct a rectangle that extends from the x-axis to any point on the curve $y = f(x)$ that is above the subinterval; the particular point does not matter–it can be above the center, above and endpoint, or above any other point in the subinterval.

(2) For each n, the total area of the rectangle can be viewed as an approximation to the exact area under the curve over the interval $[a,b]$. Moreover, it is evident intuitively that as n increases, these approximations will get better and better and will approach the exact area as a limit.

Example 1 Find the area under the curve $y = x^2$ over the interval $[2,5]$.

We begin by dividing the interval $[2,5]$ into n equal subintervals.
Each subinterval has length $\frac{3}{n}$ $\left(\text{because } \frac{5-2}{n} = \frac{3}{n}\right)$ and the endpoints occur at

$$2, 2+\frac{3}{n}, 2+\frac{6}{n}, 2+\frac{9}{n}, \ldots, 2+\frac{3(n-1)}{n}, 2+\frac{3n}{n} = 2+3 = 5$$

Over each interval, we will construct a rectangle whose height is the value of the function at some point in the interval. If we use the right-hand endpoints, the heights of the rectangles will be

$$\left(2+\frac{3}{n}\right)^2, \left(2+\frac{6}{n}\right)^2, \left(2+\frac{9}{n}\right)^2, \ldots, \left(2+\frac{3(n-1)}{n}\right)^2, 5^2$$

Since each rectangle has a base of width $\frac{3}{n}$, the total area of all of the rectangles is

$$A_n = \left[\left(2+\frac{3(1)}{n}\right)^2 + \left(2+\frac{3(2)}{n}\right)^2 + \left(2+\frac{3(3)}{n}\right)^2 + \ldots + \left(2+\frac{3(k)}{n}\right) + \ldots + \left(2+\frac{3(n-1)}{n}\right)^2 + \left(2+\frac{3(n)}{n}\right)^2\right]\left(\frac{3}{n}\right)$$

$A_n = \frac{3}{n}\sum_{k=1}^{n}\left(2+\frac{3k}{n}\right)^2$. This expression can be computed for various values of n.

Using the TI-83/84 Plus, and a particular value of n, the expression can be evaluated like this: $3/n\, *$sum(seq($(2+3*k/n)^2$,k,1,n,1))

For example, for $n = 2$, it would be

$A_2 = 3/2\, *$sum(seq($(2+3*k/2)^2$,k,1,2,1)) = 55.875

n	1	2	3	4	5	10	100	800
A_n	75	55. 875	50	47. 15625	45. 48	42. 195	39. 31545	39. 03938

The table above suggests that the area under the curve between 0 and 3 is approximately 39.
If an exact expression is desired, it goes like this:

$$
\begin{aligned}
\frac{3}{n}\sum_{k=1}^{n}\left(2+\frac{3k}{n}\right)^2 &= \frac{3}{n}\sum_{k=1}^{n}\left(4+\frac{12}{n}k+\frac{9}{n^2}k^2\right) \\
&= \frac{3}{n}\left(\sum_{k=1}^{n}4+\sum_{k=1}^{n}\frac{12}{n}k+\sum_{k=1}^{n}\frac{9}{n^2}k^2\right) \\
&= \frac{3}{n}\left(\sum_{k=1}^{n}4+\frac{12}{n}\sum_{k=1}^{n}k+\frac{9}{n^2}\sum_{k=1}^{n}k^2\right) \\
&= \frac{3}{n}\left((4n)+\frac{12}{n}\left(\frac{n(n+1)}{2}\right)+\frac{9}{n^2}\left(\frac{n(n+1)(2n+1)}{6}\right)\right) \\
&= \frac{3}{n}\left(4n+6(n+1)+\frac{3}{n}\left(\frac{(n+1)(2n+1)}{2}\right)\right) \\
&= \frac{3}{n}\left(4n+6n+6+\frac{3}{n}\left(\frac{2n^2+3n+1}{2}\right)\right) \\
&= \frac{3}{n}\left(10n+6+3n+\frac{9}{2}+\frac{3}{2n}\right) \\
&= \frac{3}{n}\left(13n+\frac{21}{2}+\frac{3}{2n}\right) \\
&= 39+\frac{63}{2n}+\frac{9}{2n^2}
\end{aligned}
$$

If we accept that the sum of the areas of the rectangles approaches
the actual area under the curve, we have this.

$$\lim_{n\to\infty}\frac{3}{n}\sum_{k=1}^{n}\left(2+\frac{3k}{n}\right)^2 = \lim_{n\to\infty}\left(39+\frac{63}{2n}+\frac{9}{2n^2}\right) = 39$$

Over each interval, we will construct a rectangle whose height is the value of the function
at some point in the interval. If we use the left-hand endpoints, the heights of the rectangles will be

$$2^2, \left(2+\frac{3}{n}\right)^2, \left(2+\frac{6}{n}\right)^2, \left(2+\frac{9}{n}\right)^2, \ldots, \left(2+\frac{3(n-1)}{n}\right)^2$$

Since each rectangle has a base of width $\frac{3}{n}$, the total area of all of the rectangles is

$$A_n = \left[2^2+\left(2+\frac{3(1)}{n}\right)^2+\left(2+\frac{3(2)}{n}\right)^2+\left(2+\frac{3(3)}{n}\right)^2+\ldots+\left(2+\frac{3(k)}{n}\right)+\ldots+\left(2+\frac{3(n-1)}{n}\right)^2\right]\left(\frac{3}{n}\right)$$

$A_n = \frac{3}{n}\sum_{k=1}^{n}\left(2+\frac{3(k-1)}{n}\right)^2$. This expression can be computed for various values of n.

Using the TI-83/84 Plus, and a particular value of n, the expression can be evaluated
like this: $3/n *$sum(seq($(2+3*(k-1)/n)^2,k,1,n,1$))
For example, for $n=2$, it would be
$A_2 = 3/2 *$sum(seq($(2+3*(k-1)/2)^2,k,1,2,1$)) $=$ 55.875

n	1	2	3	4	5	10	100	800
A_n	12	24.375	29	31.40625	32.88	35.895	38.68545	38.960632

The table above suggests that the area under the curve between 0 and 3 is approximately 39.
If an exact expression is desired, it goes like this:

$$\frac{3}{n}\sum_{k=1}^{n}\left(2+\frac{3(k-1)}{n}\right)^2 = 39-\frac{63}{2n}+\frac{9}{2n^2}$$

If we accept that the sum of the areas of the rectangles approaches
the actual area under the curve, we have this.

$$\lim_{n\to\infty}\frac{3}{n}\sum_{k=1}^{n}\left(2+\frac{3(k-1)}{n}\right)^2 = \lim_{n\to\infty}\left(39-\frac{63}{2n}+\frac{9}{2n^2}\right) = 39$$