5.3 The Definite Integral

Riemann Sum

Let f be defined on the closed interval $[a,b]$, and let Δ be a partition of $[a,b]$
given by $a = x_0 < x_1 < x_2 <\ldots< x_{n-1} < x_n = b$
where Δx_i is the width of the i^{th} subinterval $[x_{i-1}, x_i]$.
If c_i is any point in the i^{th} subinterval, then the sum

$$\sum_{i=1}^{n} f(c_i)\Delta x_i, \quad x_{i-1} \le c_i \le x_i$$

is called a **Riemann sum** of f for the partition Δ.

The width of the largest subinterval of a partition Δ is called the **norm** of the partition and is denoted by $\|\Delta\|$. If every subinterval is of equal width, the partition is said to be regular and the norm is denoted by $\|\Delta\| = \Delta x = \dfrac{b-a}{n}$.

The smaller the norm of a partition is, the more subintervals there must be that make it up.
So, $\|\Delta\| \to 0$ implies that $n \to \infty$.

If f is defined on the closed interval $[a,b]$ and the limit of Riemann sums over partitions Δ

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

exists, then f is said to be **integrable** on $[a,b]$ and the limit is denoted by

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i = \int_a^b f(x)dx.$$

The limit is called the definite integral of f from a to b. The number a is the **lower limit** of integration, and the number b is the **upper limit** of integration.

Theorem If a function is continuous on the closed interval $[a,b]$, then f is integrable on $[a,b]$.
That is, $\int_a^b f(x)dx$ exists.

In the next section we will see that antiderivatives can be used to evaluate definite integrals.
For now, however, we will evaluate them using some elementary area formulas from geometry.

Properties of the Definite Integral

Theorem. If f is integrable on three closed intervals determined by a, b, and c, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Example 1 Show using a graph that the above theorem appears to be true regardless of how a, b, and c are ordered.

Example 2 Graph and shade each indicated region. Then evaluate each definite integral.

a. $\int_0^3 (3-x)dx$

b. $\int_1^4 3dx$

c. $\int_1^5 \sqrt{4-(x-3)^2}\,dx$

Example 3 Graph and shade each indicated region. Then evaluate each definite integral.

a. $\int_{0}^{2}(2x-1)dx$

b. $\int_{-1}^{1} x^3 dx$

Special Properties of the Definite Integral

DEFINITION.

(a) If a is in the domain of f, we define $\int_{a}^{a} f(x)dx = 0$.

(b) If f is integrable on $[a, b]$, then we define $\int_{b}^{a} f(x)dx = -\int_{a}^{b} f(x)dx$

Example 4 Evaluate each definite integral.

a. $\int_{2}^{2}(3x+7)dx$

b. $\int_{2}^{0} \sqrt{4-x^2}\, dx$

THEOREM. If f and g are integrable on $[a, b]$ and if c is a constant, then cf, $f+g$, and $f-g$ are integrable on $[a, b]$ and

(a) $\int_a^b cf(x)dx = c\int_a^b f(x)dx$

(b) $\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

(c) $\int_a^b [f(x)-g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$

Example 5 Evaluate $\int_0^2 \left(3+2\sqrt{4-x^2}\right)dx$.