5.4 The Fundamental Theorem of Calculus

Theorem (The Fundamental Theorem of Calculus, Part 1). If f is continuous on $[a, b]$, and if F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Example 1 $\int_4^{10} x^2 dx =$

Example 2 $\int_2^6 (x^2 + 1)dx =$

Example 3

a. $\int_0^{\pi} \sin x dx =$

b. $\int_0^{2\pi} \sin x dx =$

Example 4

a. $\int_{\ln 2}^{\ln 5} 2e^x dx =$

b. $\int_2^5 \frac{1}{x} dx =$

Example 5 $\int_{-2}^{1} x^{-2}dx =$

Example 6

a. $\int_{4}^{4}(2x^2 - 3x + 10)dx =$

b. $\int_{4}^{2} x^3 dx =$

Example 7 $\int_{-3}^{4} |x^2 - 4|dx =$

First observe that $|x^2 - 4| = \begin{cases} x^2 - 4 & \text{if } x < -2 \\ -(x^2 - 4) & \text{if } -2 \le x \le 2 \\ x^2 - 4 & \text{if } x > 2 \end{cases}$

$$\int_{-3}^{4} |x^2 - 4|dx = \int_{-3}^{-2} |x^2 - 4|dx + \int_{-2}^{2} |x^2 - 4|dx + \int_{2}^{4} |x^2 - 4|dx$$

$$= \int_{-3}^{-2} (x^2 - 4)dx + \int_{-2}^{2} -(x^2 - 4)dx + \int_{2}^{4} (x^2 - 4)dx$$

$$= \int_{-3}^{-2} (x^2 - 4)dx - \int_{-2}^{2} (x^2 - 4)dx + \int_{2}^{4} (x^2 - 4)dx$$

$$= \left[\frac{x^3}{3} - 4x \right]_{-3}^{-2} - \left[\frac{x^3}{3} - 4x \right]_{-2}^{2} + \left[\frac{x^3}{3} - 4x \right]_{2}^{4}$$

$$= \left[\left(\tfrac{(-2)^3}{3} - 4(-2) \right) - \left(\tfrac{(-3)^3}{3} - 4(-3) \right) \right] - \left[\left(\tfrac{(2)^3}{3} - 4(2) \right) - \left(\tfrac{(-2)^3}{3} - 4(-2) \right) \right]$$
$$+ \left[\left(\tfrac{(4)^3}{3} - 4(4) \right) - \left(\tfrac{(2)^3}{3} - 4(2) \right) \right]$$

$$= \left[\frac{-8}{3} + 8 - (-9 + 12) \right] - \left[\frac{8}{3} - 8 - \left(\frac{-8}{3} + 8 \right) \right] + \left[\frac{64}{3} - 16 - \left(\frac{8}{3} - 8 \right) \right]$$

$$= \left[-\frac{8}{3} + 5 \right] - \left[\frac{8}{3} - \frac{24}{3} + \frac{8}{3} - \frac{24}{3} \right] + \left[\frac{64}{3} - \frac{48}{3} - \frac{8}{3} + \frac{24}{3} \right]$$

$$= \left[\frac{7}{3} \right] - \left[-\frac{32}{3} \right] + \left[\frac{32}{3} \right]$$

$$= \frac{7}{3} + \frac{32}{3} + \frac{32}{3} = \frac{71}{3} = 23\tfrac{2}{3}$$

Theorem (The Mean-Value Theorem for Integrals). If f is continuous on a closed interval $[a, b]$, then there is at least one number c in $[a, b]$ such that

$$\int_a^b f(x)dx = f(c)(b-a)$$

Example 8 Find the value of c in the interval interval $[2, 5]$ for the function $f(x) = x^2$ whose existence is guaranteed by the Mean-Value Theorem for integrals.

Theorem (The Fundamental Theorem of Calculus, Part 2). If f is continuous on an interval I, then f has an antiderivative on I. In particular, if a is any point in I, then the function F defined by

$$F(x) = \int_a^x f(t)dt$$

is an antiderivative of f on I; that is, $F'(x) = f(x)$ for each x in I, or in an alternative notation

$$\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$$

Example 9 Let $F(x) = \int_a^x \sin t \, dt$.

a. Find $F'(x)$ by applying FTC part 1 and differentiating the result.

b. Find $F'(x)$ by applying FTC part 2.

Example 10 Let $F(x) = \int_0^x \sqrt{\cos t} \, dt$. Find each function value indicated below.

a. $F(0)$

b. $F'(0)$

c. $F''(0)$