3.1 The Derivative and the Tangent Line Problem

Definition of a Secant Line with slope m_{sec}.

If a function f is defined on an open interval containing c,
and if $c + \Delta x$ is in this interval, then the expression

$$m_{\text{sec}} = \frac{f(c + \Delta x) - f(c)}{\Delta x}$$

is the slope of the line passing through the points $(c, f(c))$ and $(c + \Delta x, f(c + \Delta x))$. This line is called a **secant line**. The expression $\frac{f(c + \Delta x) - f(c)}{\Delta x}$ is an example of what is commonly called a **difference quotient**.

Definition of a Tangent Line with slope $m_{\tan}$.

If a function f is defined on an open interval containing c, and if the limit

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = m_{\tan}$$

exists, then the line passing through $(c, f(c))$ with slope $m_{\tan}$ is the tangent line to the graph of f at the point $(c, f(c))$.

The slope $m_{\tan}$ is also called the slope of the graph at $x = c$.

Example 1 Use the limit $\lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x}$ to find the slope of the graph of $f(x) = -3x + 7$ at the point $(2, 1)$.

Example 2

a. Find the slope of the tangent line to the graph of $f(x) = 3x^2 - 8x$ at the point $(2,-4)$.

b. Find the slope of the tangent line to the graph of $f(x) = 3x^2 - 8x$ at the point $(c, f(c))$.

c. Repeat part (a) using your answer to part (b).

Definition of Derivative

The function f' (f-prime) defined by the formula $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$ is called the **derivative of** f **at** x. The domain of f' consists of all x for which the limit exists.

The process of finding the derivative of a function is called **differentiation**.
A function is **differentiable** at x if its derivative exists at x. A function is **differentiable on an open interval** (a, b) if its derivative exists at every x in the interval.

Example 3 Use the definition of derivative to find the derivative of $f(x) = x^3 - 8x^2 + 5x$.

$$
\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^3 - 8(x+\Delta x)^2 + 5(x+\Delta x) - (x^3 - 8x^2 + 5x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(x+\Delta x)(x+\Delta x)(x+\Delta x) - 8(x+\Delta x)(x+\Delta x) + 5(x+\Delta x) - (x^3 - 8x^2 + 5x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - 8\left(x^2 + 2x\Delta x + (\Delta x)^2\right) + 5x + 5\Delta x - x^3 + 8x^2 - 5x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - 8x^2 - 16x\Delta x - 8(\Delta x)^2 + 5x + 5\Delta x - x^3 + 8x^2 - 5x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - 16x\Delta x - 8(\Delta x)^2 + 5\Delta x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\Delta x\left(3x^2 + 3x\Delta x + (\Delta x)^2 - 16x - 8\Delta x + 5\right)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left(3x^2 + 3x\Delta x + (\Delta x)^2 - 16x - 8\Delta x + 5\right) \\
&= 3x^2 + 3x(0) + (0)^2 - 16x - 8(0) + 5 \\
&= 3x^2 - 16x + 5
\end{aligned}
$$

Example 4 Use the definition of derivative to find the derivative of $f(x) = \dfrac{1}{\sqrt{x}}$.

Example 5 Use the definition of derivative to find the derivative of $f(x) = \dfrac{1}{x^2}$.

Example 6 Find the derivative with respect to x of the function $f(x) = \begin{cases} 2x - 1 & \text{if } x \le 3 \\ 4 - x & \text{if } x > 3 \end{cases}$.

THEOREM If f is differentiable at a point $x = c$, then f is also continuous at $x = c$
Observe that the derivative of f at c may be defined using the alternative difference quotient $\dfrac{f(x) - f(c)}{x - c}$ as follows: $f'(c) = \lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c}$.
Furthermore, observe that the definition of continuity at c, $\lim\limits_{x \to c} f(x) = f(c)$ is equivalent to the statement $\lim\limits_{x \to c} [f(x) - f(c)] = 0$.
Finally, to prove the theorem, suppose f is differentiable at $x = c$.

Then $\lim\limits_{x \to c} [f(x) - f(c)] = \lim\limits_{x \to c} \left[(x - c)\left(\dfrac{f(x) - f(c)}{x - c} \right) \right]$

$= \left[\lim\limits_{x \to c} (x - c) \right] \left[\lim\limits_{x \to c} \left(\dfrac{f(x) - f(c)}{x - c} \right) \right]$

$= [0][f'(c)]$ This is the step that requires differentiability at c.

$= 0$

Therefore, f is continuous at $x = c$.

So, if a function has a discontinuity at $x = c$, it must also fail to be differentiable there, but it may fail to be differentiable at $x = c$ even if it is continuous.
The most common situations in which points of nondifferentiability are encountered are *corners*, *points of vertical tangency*, and *points of discontinuity*.

Example 7 Find the derivative with respect to x of the function $f(x) = |x|$.

Example 8 Find the derivative of $f(x) = x^{1/3}$ or $f(x) = \sqrt[3]{x}$.

$$
\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\sqrt[3]{x+\Delta x} - \sqrt[3]{x}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(\sqrt[3]{x+\Delta x} - \sqrt[3]{x})\left((\sqrt[3]{x+\Delta x})^2 + \sqrt[3]{x+\Delta x}\,\sqrt[3]{x} + (\sqrt[3]{x})^2\right)}{\Delta x\left((\sqrt[3]{x+\Delta x})^2 + \sqrt[3]{x+\Delta x}\,\sqrt[3]{x} + (\sqrt[3]{x})^2\right)} \\
&= \lim_{\Delta x \to 0} \frac{(\sqrt[3]{x+\Delta x})^3 - (\sqrt[3]{x})^3}{\Delta x\left((\sqrt[3]{x+\Delta x})^2 + \sqrt[3]{x+\Delta x}\,\sqrt[3]{x} + (\sqrt[3]{x})^2\right)} \\
&= \lim_{\Delta x \to 0} \frac{x + \Delta x - x}{\Delta x\left((\sqrt[3]{x+\Delta x})^2 + \sqrt[3]{x+\Delta x}\,\sqrt[3]{x} + (\sqrt[3]{x})^2\right)} \\
&= \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x\left((\sqrt[3]{x+\Delta x})^2 + \sqrt[3]{x+\Delta x}\,\sqrt[3]{x} + (\sqrt[3]{x})^2\right)} \\
&= \lim_{\Delta x \to 0} \frac{1}{\left((\sqrt[3]{x+\Delta x})^2 + \sqrt[3]{x+\Delta x}\,\sqrt[3]{x} + (\sqrt[3]{x})^2\right)} \\
&= \frac{1}{\left((\sqrt[3]{x})^2 + \sqrt[3]{x}\,\sqrt[3]{x} + (\sqrt[3]{x})^2\right)} \\
&= \frac{1}{3(\sqrt[3]{x})^2}
\end{aligned}
$$

So, $f'(x) = \dfrac{1}{3x^{2/3}}$.

Observe that 0 is not in the domain of $f'(x)$.

$f(x) = \sqrt[3]{x}$ is continuous at 0, but not differentiable at 0.