4.3 Increasing and Decreasing Functions and the First Derivative Test

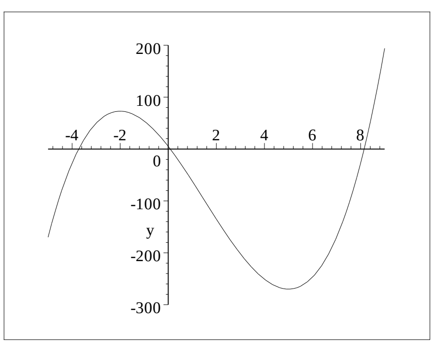
DEFINITION. Let f be defined on an interval, and let x_1 and x_2 denote points in that interval.
(a) f is **increasing** on the interval if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.
(b) f is **decreasing** on the interval if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.
(c) f is **constant** on the interval if $f(x_1) = f(x_2)$ for all points x_1 and x_2.

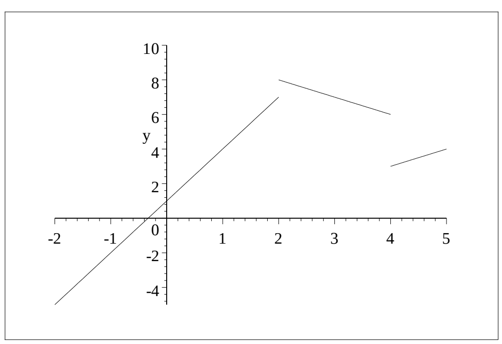
THEOREM. Let f be a function that is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b).
(a) If $f'(x) > 0$ for every value of x in (a, b), then f is increasing on $[a, b]$.
(b) If $f'(x) < 0$ for every value of x in (a, b), then f is decreasing on $[a, b]$.
(c) If $f'(x) = 0$ for every value of x in (a, b), then f is constant on $[a, b]$.

Example 1 Find the intervals over which each function is increasing and decreasing.

a. $f(x) = 2x^3 - 9x^2 - 60x + 5$

b. $f(x) = \begin{cases} 3x+1 & \text{if } x < 2 \\ -x+10 & \text{if } 2 \le x \le 4 \\ x-1 & \text{if } x > 4 \end{cases}$

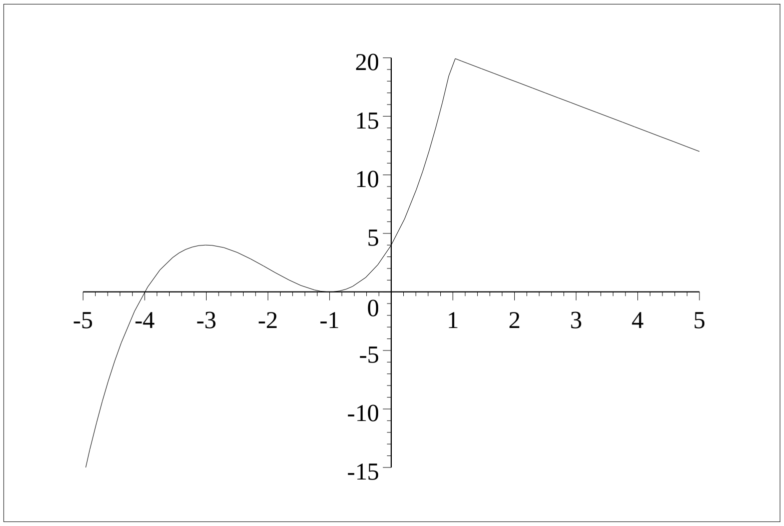




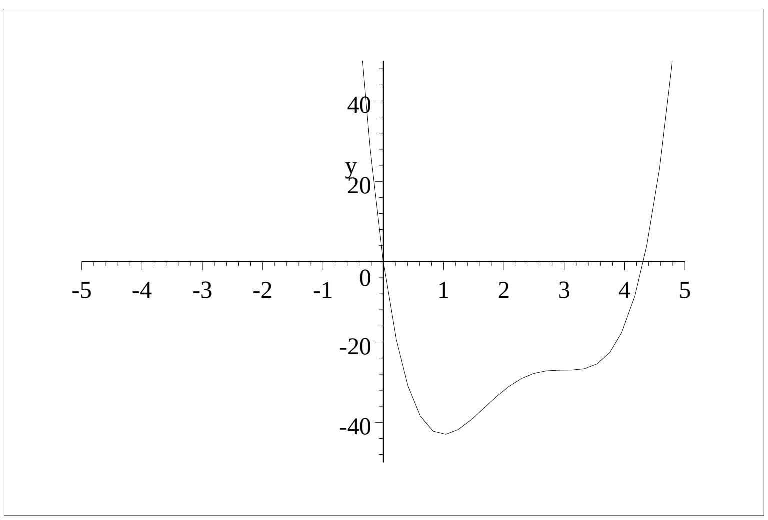
THEOREM (First Derivative Test) Suppose f is continuous at a critical point x_0.
(a) If $f'(x) > 0$ on an open interval extending left from x_0 and $f'(x) < 0$ on an open interval extending right from x_0, then f has a relative maximum at x_0.
(b) If $f'(x) < 0$ on an open interval extending left from x_0 and $f'(x) > 0$ on an open interval extending right from x_0, then f has a relative minimum at x_0.
(c) If $f'(x)$ has the same sign [either $f'(x) > 0$ or $f'(x) < 0$] on an open interval extending left from x_0 and on an open interval extending right from x_0, then f does not have a relative extremum at x_0.

Example 2 Analyze the relative extrema of $f(x) = \begin{cases} (x+4)(x+1)^2 & \textit{if} \quad x \le 1 \\ -2x+22 & \textit{if} \quad x > 1 \end{cases}$

using the First Derivative Test.

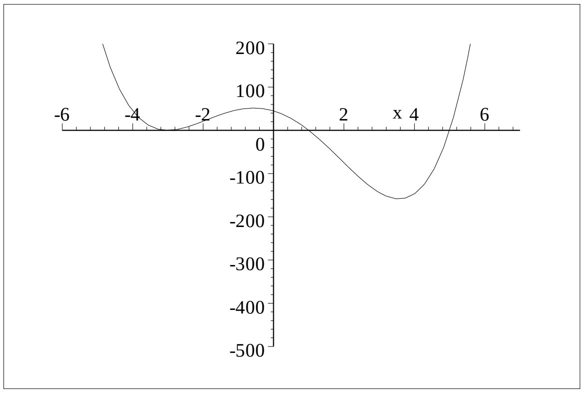


Example 3 Analyze the relative extrema of $f(x) = 3x^4 - 28x^3 + 90x^2 - 108x$ using the First Derivative Test.

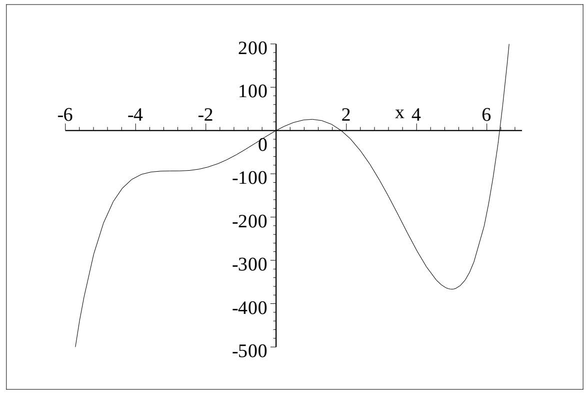


$f'(x) = 12x^3 - 84x^2 + 180x - 108 = 12(x^3 - 7x^2 + 15x - 9) = 12(x-3)^2(x-1)$

Example 4 Use the graph of *f'* to (a) identify the critical numbers of *f*, and (b) determine whether *f* has a relative maximum, a relative minimum, or neither at each critical number. Finally, (c) do your best to sketch a rough graph of a function *f* which would have this graph as its derivative.



$$f'(x) = x^4 - 22x^2 - 24x + 45$$



$$f(x) = \frac{1}{5}x^5 - \frac{22}{3}x^3 - 12x^2 + 45x$$

	critical number	relative maximum	relative minimum	neither
smallest value	9a	○	○	○ 9b
	9c	○	○	○ 9d
largest value	9e	○	○	○ 9f