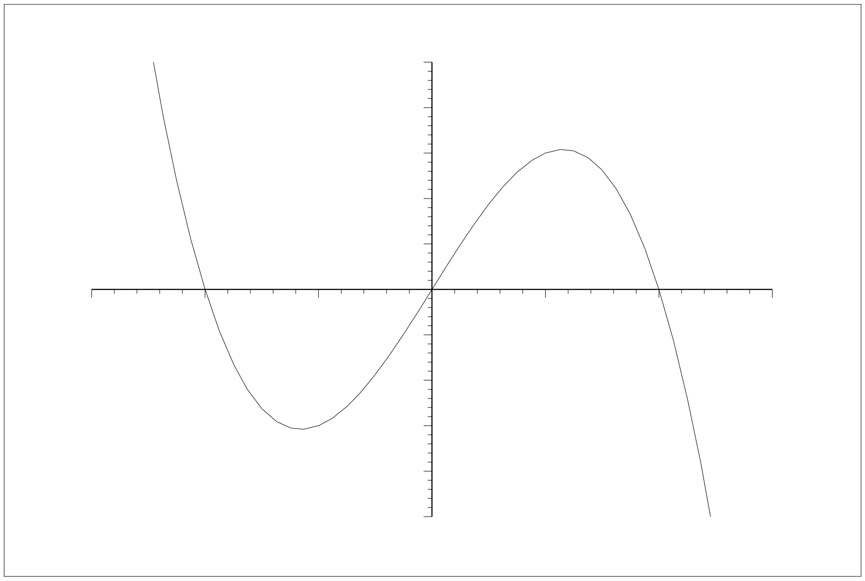
4.4 Concavity and the Second Derivative Test

Concavity



DEFINITION. If f is differentiable on an open interval I, then f is said to be **concave up** on I if f' is increasing on I, and f is said to be **concave down** on I if f' is decreasing on I.

A determination of whether f' is increasing or decreasing can be made by analyzing the sign of f''.

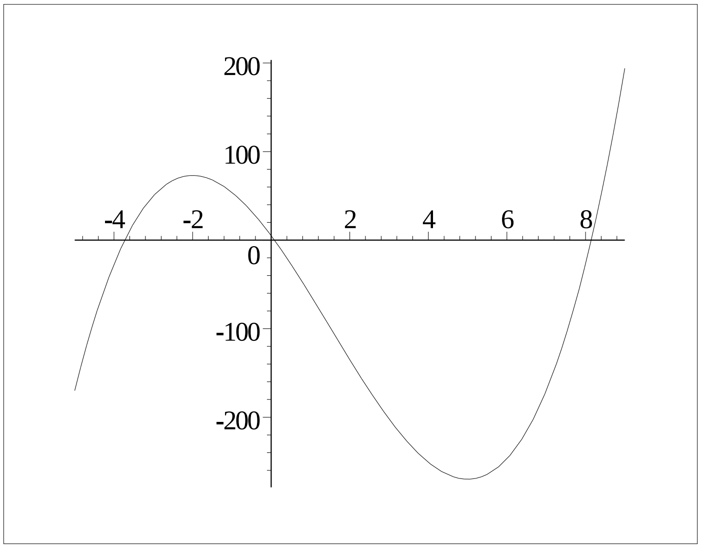
THEOREM. Let f be twice differentiable on an open interval I.
(a) If $f''(x) > 0$ on I, then f is concave up on I.
(b) If $f''(x) < 0$ on I, then f is concave down on I.

Example 1 Find open intervals on which the following functions are concave up and open intervals on which they are concave down.

(a) $f(x) = 2x - 5$

(b) $f(x) = -3x^2 - 6x + 2$

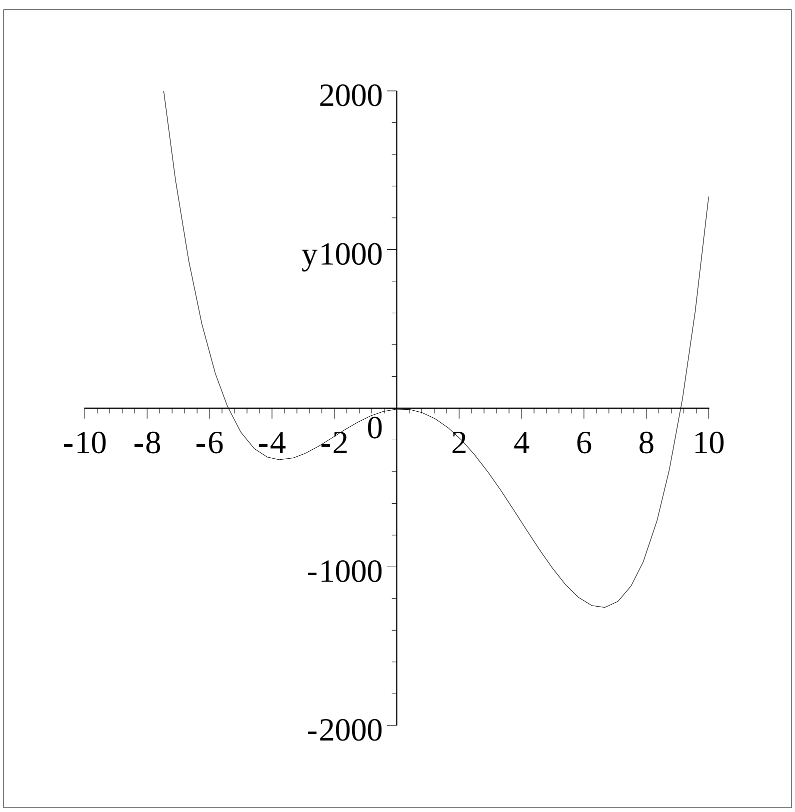
(c) $f(x) = 2x^3 - 9x^2 - 60x + 5$



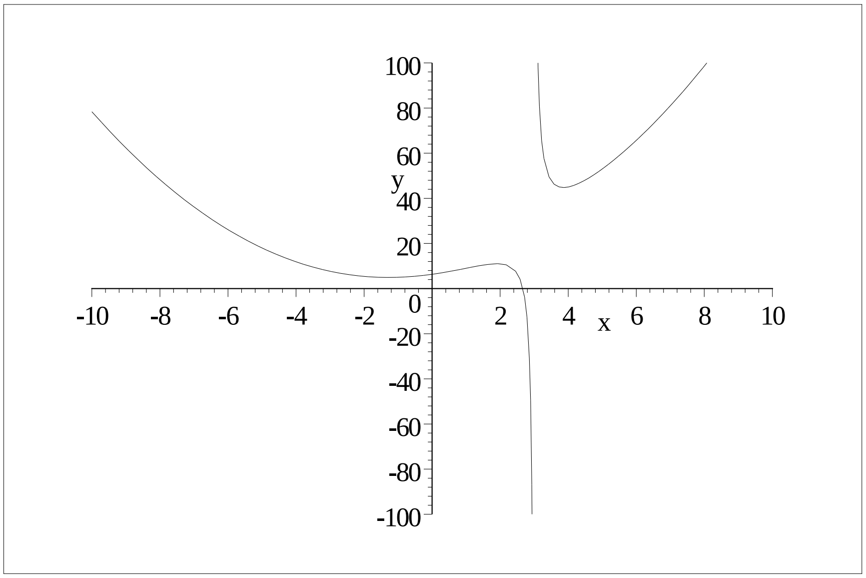
Inflection Points

DEFINITION. If f is continuous on an open interval containing the point x_0, and if f changes the direction of its concavity at that point, then we say that f has an inflection point at x_0 and we call the point $(x_0, f(x_0))$ on the graph of f an **inflection point** of f.

Example 2 Find the inflection points of $f(x) = x^4 - 4x^3 - 48x^2 + 14x - 6$.



Example 3 Find the inflection points of $f(x) = \dfrac{x^3 - 19}{x - 3} = x^2 + 3x + 9 + \dfrac{8}{x - 3}$



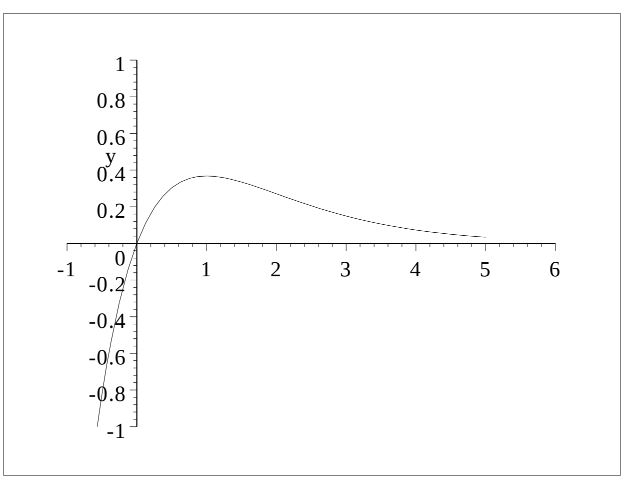
$$f'(x) = \frac{2x^3 - 9x^2 + 19}{(x-3)^2} = 2x + 3 - \frac{8}{(x-3)^2}$$

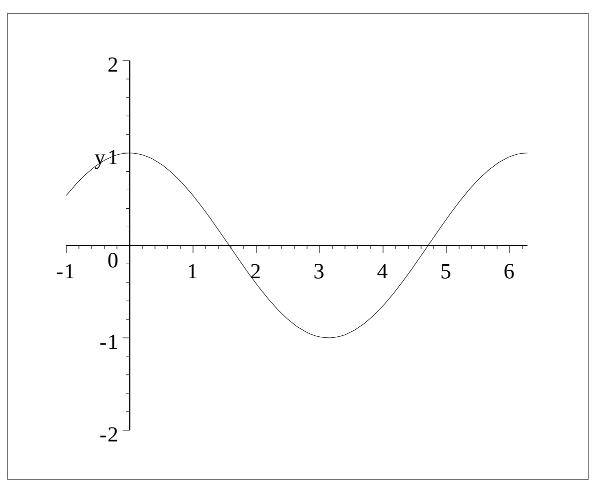
$$f''(x) = \frac{2(x^3 - 9x^2 + 27x - 19)}{(x-3)^3} = \frac{2(x-1)(x^2 - 8x + 19)}{(x-3)^3} = 2 + \frac{16}{(x-3)^3}$$

Example 4 Find the inflection points of each of the following functions.

a. $f(x) = xe^{-x}$

b. $f(x) = \cos x$





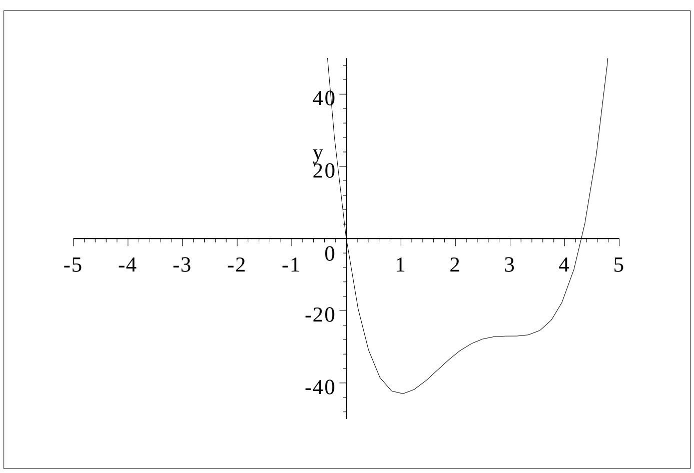
THEOREM (Second Derivative Test). Suppose that f is twice differentiable at the point x_0.

(a) If $f'(x_0) = 0$ and $f''(x_0) > 0$ then f has a relative minimum at x_0.

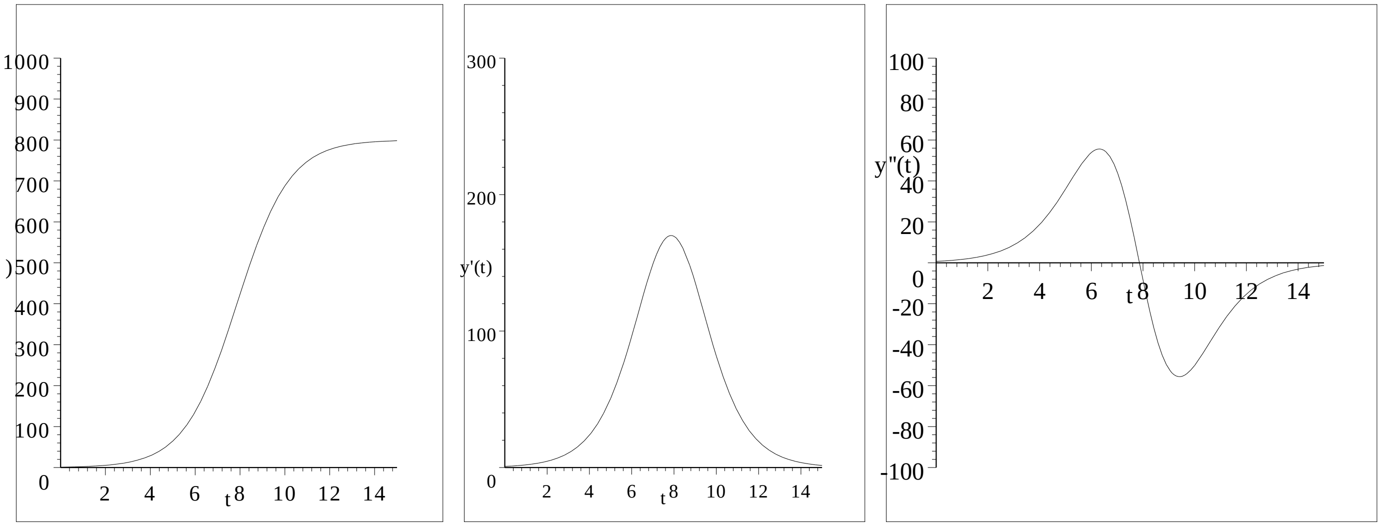
(b) If $f'(x_0) = 0$ and $f''(x_0) < 0$ then f has a relative maximum at x_0.

(c) If $f'(x_0) = 0$ and $f''(x_0) = 0$ then the test inconclusive; that is, f may have a relative maximum, a relative minimum, or neither at x_0.

Example 5 Use the second derivative test to analyze the critical points of $f(x) = 3x^4 - 28x^3 + 90x^2 - 108x$.



Example 6 Suppose that the number of individuals on a secluded college campus who have been infected by a flu virus is modeled by the function $y(t) = \dfrac{800}{1 + 799e^{-0.85t}}$ where $y(t)$ is the number of infected students at time t (in days, starting with $t = 0$). On what day is the virus spreading the most rapidly.



$$y(t) = \frac{800}{1 + 799e^{-0.85t}} \qquad y'(t) = \frac{543320e^{-0.85t}}{(1 + 799e^{-0.85t})^2} \qquad y''(t) = \frac{461820(799e^{-1.7t} - e^{-0.85t})}{(1 + 799e^{-0.85t})^3}$$

t	$y(t)$	$y'(t)$	$y''(t)$
0	1	0.84894	0.71979
1	2.3357	1.9796	1.6728
2	5.4435	4.5955	3.853
3	12.621	10.558	8.6915
4	28.917	23.691	18.682
5	64.531	50.427	35.948
6	136.26	96.092	53.855
7	259.57	149.05	44.478
8	423.3	169.42	−8.389
9	579.56	135.74	−51.794
10	688.13	81.792	−50.079
11	748.02	41.31	−30.55
12	776.93	19.047	−15.256
13	789.97	8.4168	−6.9749
14	795.68	3.6497	−3.0687
15	798.15	1.5696	−1.328