3.6 Derivatives of Inverse Functions

Theorem. Suppose that the domain of a function f is an open interval I and that f is differentiable and one-to-one on this interval. Then f^{-1} is differentiable at any point x in the range of f at which $f'(f^{-1}(x)) \neq 0$, and its derivative is $\frac{d}{dx}[f^{-1}(x)] = \frac{1}{f'(f^{-1}(x))}$

If $y = f^{-1}(x)$, then $x = f(y)$ and $\frac{dy}{dx} = \frac{d}{dx}[f^{-1}(x)]$ and $\frac{dx}{dy} = f'(y) = f'(f^{-1}(x))$.

So, from the theorem above, we can write

$$\frac{dy}{dx} = \frac{d}{dx}[f^{-1}(x)] = \frac{1}{f'(f^{-1}(x))} = \frac{1}{\frac{dx}{dy}}.$$

This is sometimes a useful alternative version of the above theorem.

Example 1 Confirm the formula $\frac{dy}{dx} = \frac{1}{dx/dy}$ for the function $f(x) = x^3 - 5$.

$$y = x^3 - 5 \qquad \frac{dy}{dx} = 3x^2$$

$$x^3 = y + 5$$

$$x = \sqrt[3]{y+5}$$

$$\frac{dx}{dy} = \frac{1}{3}(y+5)^{-2/3}(1)$$

$$\frac{dx}{dy} = \frac{1}{3(y+5)^{2/3}}$$

$$\frac{dx}{dy} = \frac{1}{3\left(\sqrt[3]{y+5}\right)^2}$$

$$\frac{dx}{dy} = \frac{1}{3x^2}$$

Therefore $\frac{dy}{dx}$ and $\frac{dx}{dy}$ are in fact reciprocals of each another.

Theorem. Suppose that the domain of a function f is an open interval I on which $f'(x) > 0$ or on which $f'(x) < 0$. Then f is one-to-one, $f^{-1}(x)$ is differentiable at all values of x in the range of f, and the derivative of $f^{-1}(x)$ is given by $\frac{d}{dx}[f^{-1}(x)] = \frac{1}{f'(f^{-1}(x))}$.

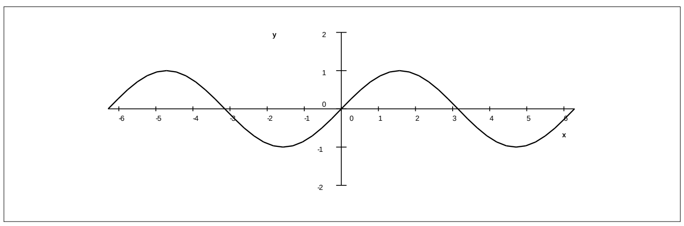
Example 2 Consider the function $f(x) = x^3 + 3x + 2$.

a. Show that f is one-to-one on the interval $(-\infty, \infty)$.

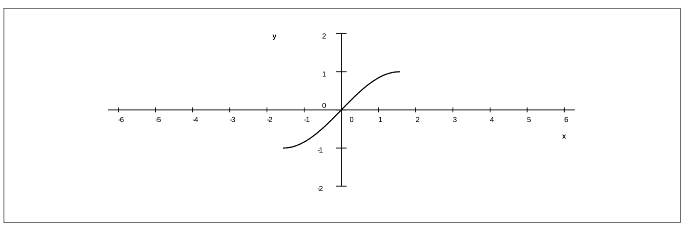
b. Find a formula for the derivative of f^{-1} using $\frac{dy}{dx} = \frac{1}{dx/dy}$.

All of the six familiar trigonometric functions are periodic and, hence, fail the horizontal line test pretty badly. The first step toward finding inverses for these functions is to restrict the domain of each function suitably so that the range of each is preserved and so that each function is one-to-one with its restricted domain.

We will find inverses only for the functions $\sin x$, $\cos x$, $\tan x$, and $\sec x$ in this section.
When deciding on a restriction for each trigonometric function, we will take the following factors into consideration.
(1) The range of the function with restricted domain must be the same as that of the original function.
(2) The interval $0 < x < \frac{\pi}{2}$ (the first quadrant) must be in the restricted domain.
(3) The function with restricted domain should be as continuous as possible.



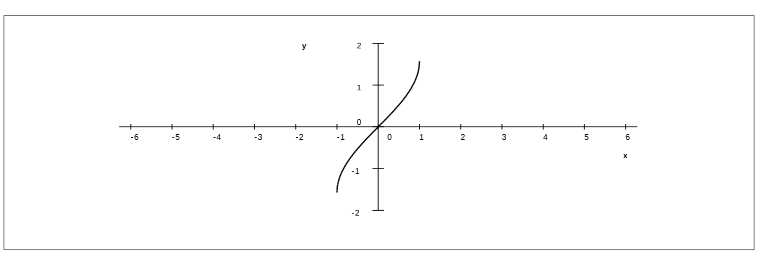
$y = \sin x$



$y = \sin x$ (with restricted domain)

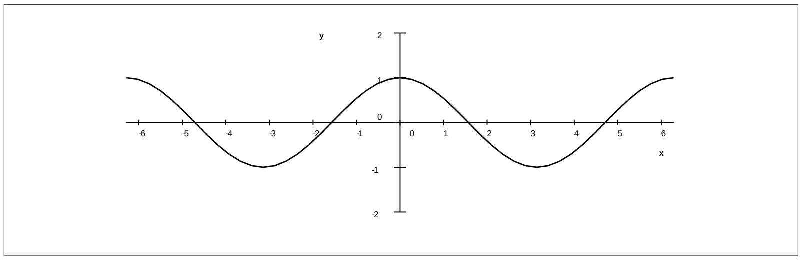
The range of the unrestricted sine function is $[-1, 1]$.
$\sin x$ takes on only the values in the interval $[0, 1]$ in the first quadrant, so an adjacent quadrant must be added.
The second quadrant contributes nothing new, for $\sin x$ takes on only the values in the interval $[0, 1]$ in the second quadrant as well.
The fourth quadrant works much better, because $\sin x$ takes on the values in the interval $[-1, 0]$ there, so $\sin x$ takes on all values in the interval $[-1, 1]$ in the union of the first and fourth quadrants $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

DEFINITION. $\arcsin x$ is the inverse of the restricted sine function
$\sin x,\ -\frac{\pi}{2} \le x \le \frac{\pi}{2}$.
The domain of $\arcsin x$ is $-1 \le x \le 1$. The range of $\arcsin x$ is $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

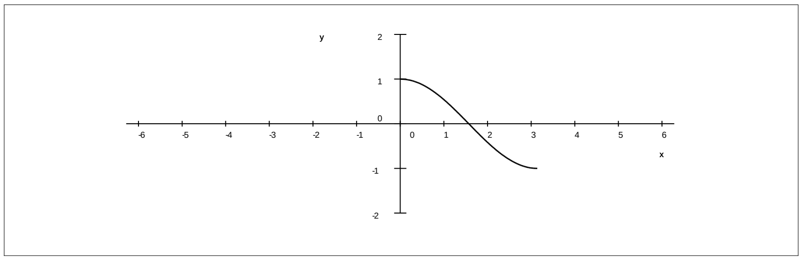


$y = \arcsin x$

$y = \cos x$



$y = \cos x$



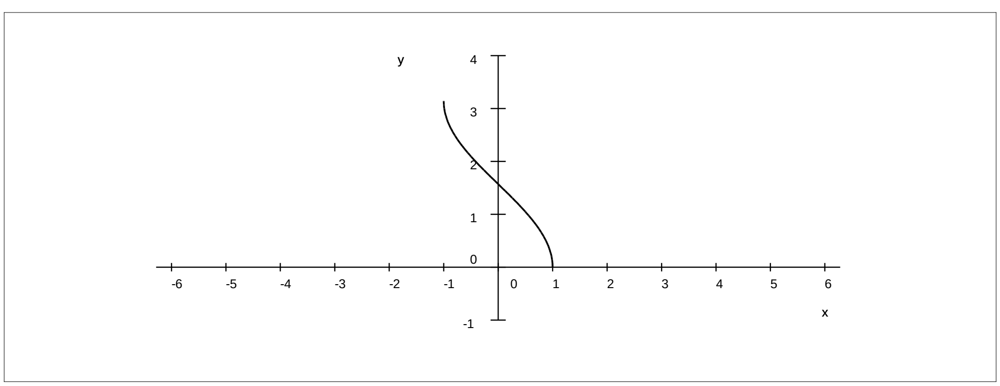
$y = \cos x$ (with restricted domain)

The range of the unrestricted cosine function is $[-1, 1]$.
$\cos x$ takes on only the values in the interval $[0, 1]$ in the first quadrant, so an adjacent quadrant must be added.
The fourth quadrant contributes nothing new, because $\cos x$ takes on only the values in the interval $[0,\ 1]$ in the fourth quadrant as well.
The second quadrant works much better, because $\cos x$ takes on the values in the interval $[-1, 0]$ there. So $\cos x$ takes on all values in the interval $[-1,\ 1]$ in the union of the first and second quadrants $(0 \le x \le \pi)$.

DEFINITION. $\arccos x$ is the inverse of the restricted cosine function

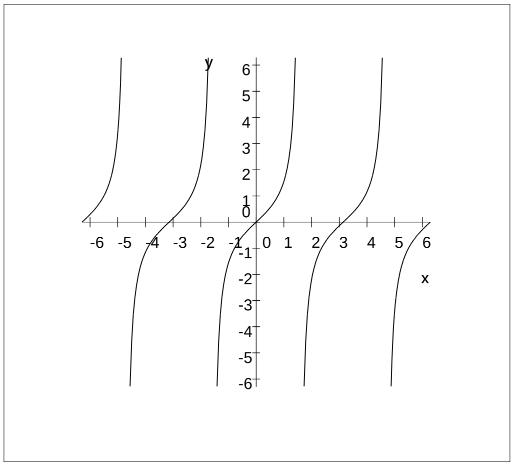
$$\cos x,\ 0 \le x \le \pi.$$

The domain of $\arccos x$ is $-1 \le x \le 1$. The range of $\arccos x$ is $0 \le x \le \pi$.

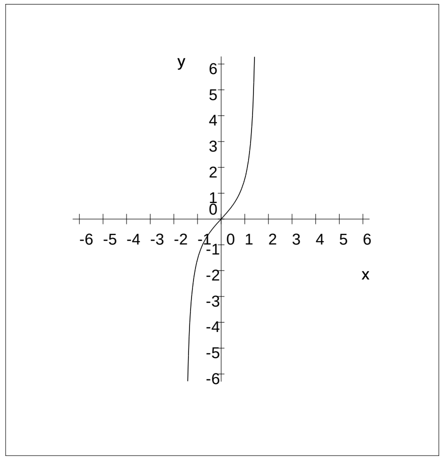


$y = \arccos x$

$y = \tan x$



$y = \tan x$



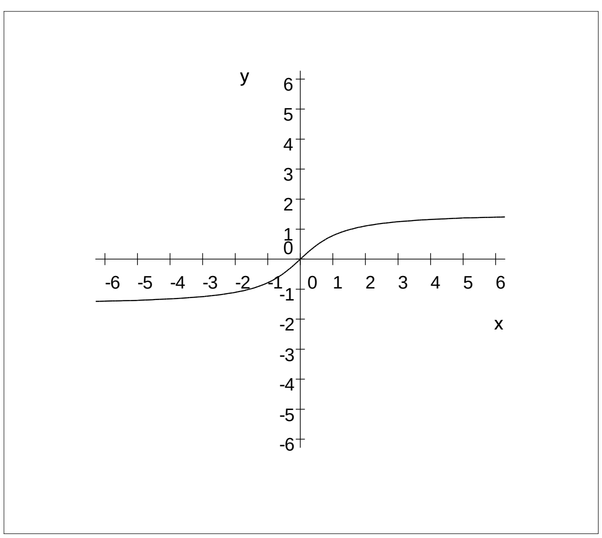
$y = \tan x$ (with restricted domain)

The range of the unrestricted tangent function is $(-\infty, +\infty)$.
$\tan x$ takes on only the values in the interval $[0, +\infty)$ in the first quadrant,
so an adjacent quadrant must be added.
The second quadrant contributes the remaining range that we need $(-\infty, 0]$,
but since $\tan x$ is undefined and has a vertical asymptote at $\frac{\pi}{2}$, quadrant II
proves to be somewhat undesirable.
It is better to restrict the domain of $\tan x$ to the union of the first and fourth
quadrants, because $\tan x$ is continuous on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and takes on its
entire range there as well.

DEFINITION. $\arctan x$ is the inverse of the restricted tangent function

$\tan x,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$.

The domain of $\arctan x$ is $-\infty < x < \infty$. The range of $\arctan x$ is $-\frac{\pi}{2} < x < \frac{\pi}{2}$.



$y = \arctan x$

Example 3 Evaluate each inverse trigonometric function as indicated below.

a. $\arcsin\left(-\frac{1}{2}\right)$ b. $\arccos\left(-\frac{\sqrt{3}}{2}\right)$ c. $\arctan(-1)$ d. $\operatorname{arcsec}\left(-\sqrt{2}\right)$

Example 4 Evaluate each composition.

a. $\arcsin\left(\sin\left(\frac{3\pi}{4}\right)\right)$ b. $\arccos\left(\cos\left(\frac{3\pi}{4}\right)\right)$ c. $\arctan\left(\tan\left(\frac{3\pi}{4}\right)\right)$

Example 5 Verify each of the following derivatives.

$\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}}$ $\frac{d}{dx}[\arccos x] = \frac{-1}{\sqrt{1-x^2}}$

Example 6 Verify the following derivatives.

$$\frac{d}{dx}[\arctan x] = \frac{1}{1+x^2}$$

$$\frac{d}{dx}[\operatorname{arcsec} x] = \frac{1}{|x|\sqrt{x^2-1}}$$

Let u be any differentiable function of x.

$$\frac{d}{dx}[\sin^{-1} u] = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$

$$\frac{d}{dx}[\cos^{-1} u] = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}$$

$$\frac{d}{dx}[\tan^{-1} u] = \frac{1}{1+u^2}\frac{du}{dx}$$

$$\frac{d}{dx}[\cot^{-1} u] = \frac{-1}{1+u^2}\frac{du}{dx}$$

$$\frac{d}{dx}[\sec^{-1} u] = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$$

$$\frac{d}{dx}[\csc^{-1} u] = \frac{-1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$$

Example 7 Find $\frac{d}{dx}\left[\arctan\left(\frac{1}{x}\right)\right]$ given that $\frac{d}{dx}[\arctan u] = \frac{1}{1+u^2}\frac{du}{dx}$.