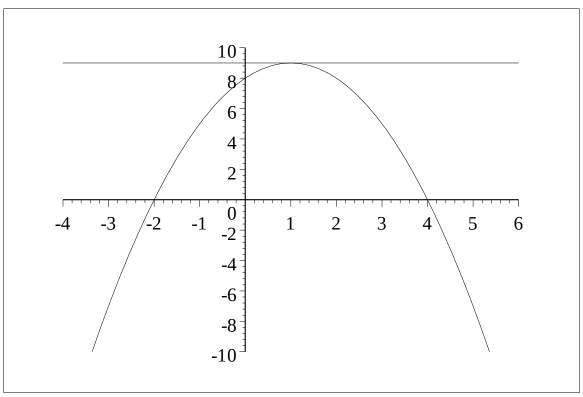
4.2 Rolle's Theorem and the Mean Value Theorem

Rolle's Theorem

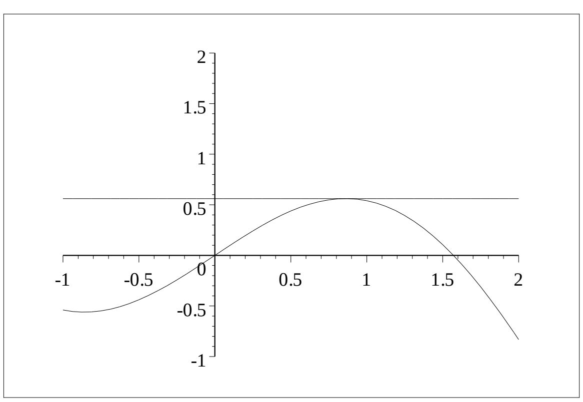
THEOREM (Rolle's Theorem). Let f be differetiable on (a,b) and continuous on $[a,b]$. If $f(a) = f(b)$, then there is at least one point c in (a,b) where $f'(c) = 0$.

Spotted in Beijing - The Mean Value Theorem

Example 1 Let $f(x) = -x^2 + 2x + 8$. Observe that $f(-3) = -7$ and $f(5) = -7$. Find a number c in the interval $(-3, 5)$ whose existence is guaranteed by Rolle's Theorem.



Example 2 Use Rolle's Theorem to show that the graph of $f(x) = x\cos x$ has a horizontal tangent line in the interval $\left[0, \frac{\pi}{2}\right]$.



$f(x) = x\cos x$

The Mean-Value Theorem

Rolle's Theorem is a special case of the Mean-Value Theorem, which can be obtained from Rolle's Theorem by "tilting" it a bit.

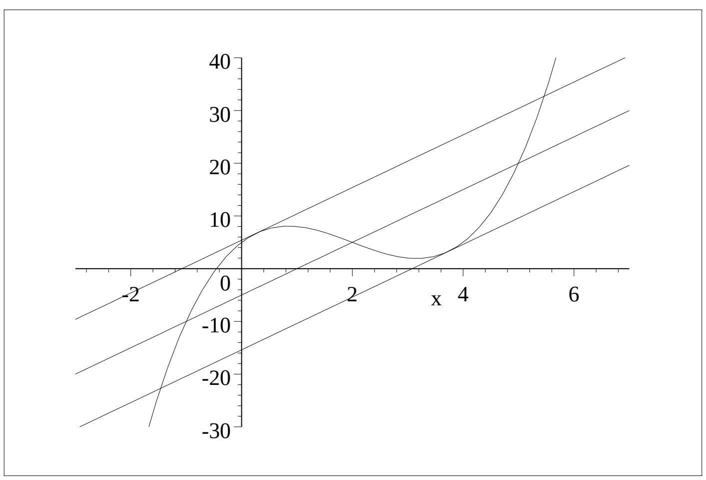
THEOREM (Mean-Value Theorem). Let f be differentiable on (a,b) and continuous on $[a,b]$. Then there is at least one number c in (a,b) where

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

Note that $m = \dfrac{f(b)-f(a)}{b-a}$ is the slope of the secant line joining the points $(a,f(a))$ and $(b,f(b))$ on the graph of f. The number m is the average rate of change of the function between a and b.

The Mean-Value Theorem literally says that subject to certain conditions (differentiability, continuity, etc.), the average rate of change of a function $\dfrac{f(b)-f(a)}{b-a}$ on an interval (a,b) is equal to the instantaneous rate of change of the function $f'(c)$ at at least one point in that interval.

Example 3 Let $f(x) = x^3 - 6x^2 + 8x + 5$. Show that f satisfies the hypotheses of the Mean-Value Theorem on the interval $[-1,5]$. Find the values of c in the interval $(-1,5)$ whose existence is guaranteed by the Mean-Value Theorem.



$$f'(c) = \frac{f(5)-f(-1)}{5-(-1)}$$

$$3c^2 - 12c + 8 = \frac{20-(-10)}{6}$$

$$3c^2 - 12c + 8 = 5$$

$$3c^2 - 12c + 3 = 0$$

$$c = 2+\sqrt{3},\ c = 2-\sqrt{3}$$

Example 4

a. Use the MVT to show that if f is differentiable on an open interval I, and if $|f'(x)| \le 1$ for all x in I, then $|f(x) - f(y)| \le |x - y|$ for all x and y in I.

Let f be a differentiable function on an open interval I and suppose $|f'(x)| \le 1$.
Let x and y be any two distinct numbers in the interval I, and without loss of generality assume $y < x$.
Since f is differentiable on the interval I, it therefore continuous at y and x and differentiable on the interval (y, x) and thus satisfies the hypotheses of the MVT on the interval $[y, x]$.
By the MVT, there exists a number c in I such that $f'(c) = \dfrac{f(x) - f(y)}{x - y}$.
We then have the following sequence of equations and inequalities.

$$\frac{|f(x) - f(y)|}{|x - y|} = \left|\frac{f(x) - f(y)}{x - y}\right| = |f'(c)| \le 1$$

Multiplying each statement by $|x - y|$, we obtain the statement:

$$|f(x) - f(y)| \le |x - y|$$

b. Conclude that $|\sin x - \sin y| \le |x - y|$ and $|\cos x - \cos y| \le |x - y|$ for all x and y.

Since $\left|\dfrac{d}{dx}[\sin x]\right| = |\cos x| \le 1$ and $\left|\dfrac{d}{dx}[\cos x]\right| = |-\sin x| = |\sin x| \le 1$,
these statements hold for the sine and cosine functions.