3.2 Basic Differentiation Rules and Rates of Change

While it is important for us to understand the procedure for finding the derivative of a function using the definition $f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$, we will begin in this section to develop time-saving techniques for finding derivatives quickly and efficiently without the definition.

First, a few words about notation.
There are several different ways we can denote the derivative of f.
In the previous section, we used the limit definition to show that the derivative of $f(x) = 3x^2 + 17$ is given by $f'(x) = 6x$. Since a function can be uniquely defined by an algebraic expression, we may refer to the derivative of the expression which defines the function.
The notation $\frac{d}{dx}[3x^2 + 17] = 6x$ means "the derivative of $3x^2 + 17$ with respect to x equals $6x$."
We say it like this, "*dee-dee-ecks* of $3x^2 + 17$ equals $6x$."
If $y = 3x^2 + 17$, we write $\frac{dy}{dx} = 6x$.
We say it like this, "*dee-wye-dee-ecks* equals $6x$."

THEOREM The Constant Rule
The derivative of a constant function is 0; that is, if c is any real number, then $\frac{d}{dx}[c] = 0$.

Example 1 If $f(x) = -3$, then $f'(x) = 0$.

THEOREM The Power Rule

If $n \neq 0$, then $\frac{d}{dx}[x^n] = nx^{n-1}$.

Example 2 Find the derivative of each function

a. $f(x) = x^{35}$

b. $g(x) = \sqrt[3]{x^2}$

c. $h(x) = \dfrac{1}{x^4}$

THEOREM The Constant Multiple Rule

If f is differentiable at x and c is any real number, then cf is also differentiable at x and $\dfrac{d}{dx}[cf(x)] = c\dfrac{d}{dx}[f(x)]$.

In words, *the derivative of a constant times a function is the constant times the derivative of the function*.

Example 3 Find the derivative of each function.

a. $y = 3x^{-7}$

b. $y = -8x^4$

c. $y = \dfrac{5}{3}\sqrt{x}$

Example 4 Find the derivative of each function. Be careful with the parentheses!

a. $y = \dfrac{7}{4x^3}$

b. $y = \dfrac{7}{(4x)^3}$

c. $y = \dfrac{7}{4x^{-3}}$

d. $y = \dfrac{7}{(4x)^{-3}}$

THEOREM The Sum and Difference Rules

If f and g are differentiable at x, then so are $f+g$ and $f-g$ and

$$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)] = f'(x)+g'(x)$$

$$\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)] = f'(x)-g'(x)$$

In words, *the derivative of the sum is equal to the sum of the derivatives*.
Also, *the derivative of the difference is equal to the difference of the derivatives*.

Example 5 Let $f(x) = 3x^4 + 6x^2$. Find $f'(x)$.

Recall the following limits from Chapter 2.

$$\lim_{\Delta x\to 0} \frac{\sin \Delta x}{\Delta x} = 1 \qquad \textit{and} \qquad \lim_{\Delta x\to 0} \frac{1-\cos(\Delta x)}{\Delta x} = 0$$

Also recall the following addition formulas from trigonometry:

$\sin(a+b) = \sin a \cos b + \cos a \sin b$

$\cos(a+b) = \cos a \cos b - \sin a \sin b$

Use the definition of the derivative and find $\frac{d}{dx}[\sin x]$ and $\frac{d}{dx}[\cos x]$.

$$\begin{aligned}
\frac{d}{dx}[\sin x] &= \lim_{\Delta x\to 0} \frac{\sin(x+\Delta x)-\sin x}{\Delta x} \\
&= \lim_{\Delta x\to 0} \frac{\sin(x)\cos(\Delta x)+\cos x\sin(\Delta x)-\sin x}{\Delta x} \\
&= \lim_{\Delta x\to 0} \left[\frac{(\sin x)(\cos(\Delta x)-1)+\cos(x)\sin(\Delta x)}{\Delta x} \right] \\
&= \lim_{\Delta x\to 0} \left[\frac{(\sin x)(\cos(\Delta x)-1)}{\Delta x} + \frac{\cos(x)\sin(\Delta x)}{\Delta x} \right] \\
&= \lim_{\Delta x\to 0} \left[-\sin x \left(\frac{1-\cos(\Delta x)}{\Delta x} \right) + \cos(x) \left(\frac{\sin(\Delta x)}{\Delta x} \right) \right] \\
&= \lim_{\Delta x\to 0} (-\sin x) \cdot \lim_{\Delta x\to 0} \left(\frac{1-\cos(\Delta x)}{\Delta x} \right) + \lim_{\Delta x\to 0} \cos(x) \cdot \lim_{\Delta x\to 0} \left(\frac{\sin(\Delta x)}{\Delta x} \right) \\
&= \quad (-\sin x) \quad \cdot \quad (0) \quad + \quad (\cos x) \quad \cdot \quad (1) \\
&= \cos x
\end{aligned}$$

Now, show that $\frac{d}{dx}[\cos x] = -\sin x$.

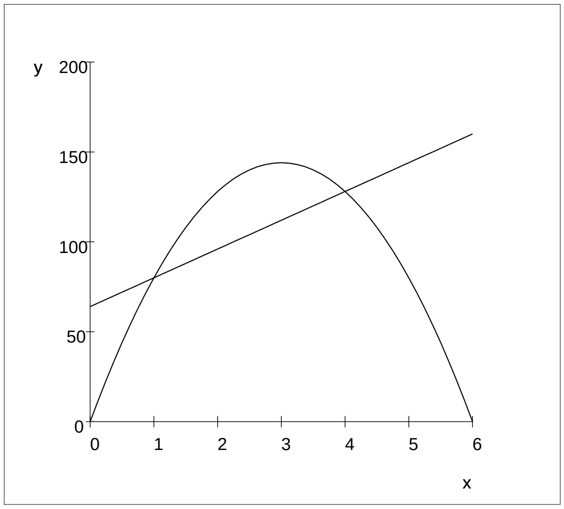
Rectilinear Motion

If a particle travels at a constant velocity of 60 mph in a given direction, the function $s(t) = 60t$ indicates the distance s (in miles) that the particle has traveled at time t (in hours). This is a linear function, and the slope of the graph $m = 60$ mi/hr represents the particle's velocity at any given time.

If the position versus time function $s = f(t)$ for a particular particle is not linear, the average velocity v_{ave} of the particle over a particular time interval $[t,\ t + \Delta t]$ can be found by dividing the change in the particle's position $\Delta s = f(t + \Delta t) - f(t)$ during that time interval by the time elapsed: $t + \Delta t - t = \Delta t$.

$$v_{ave} = \frac{\text{change in distance}}{\text{change in time}} = \frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t}$$

Example 6 If the position of a particle is given by the function $s(t) = -16t^2 + 96t,$ find and compare the particle's average velocity and the average speed on the time interval $1 \le t \le 4$.



If the position function of a particle is given by $s = f(t),$ the **instantaneous velocity** at the point $(t, f(t))$ is given by

$$v(t) = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} = s'(t).$$

Example 7 Find the instantaneous velocity and instantaneous speed of the particle in Example 6 at each time

a. $t = 1$ b. $t = 2.5$ c. $t = 4.$