4.5 Limits at Infinity

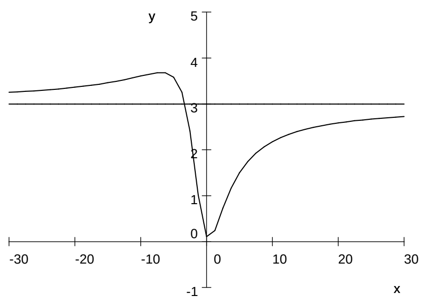
The end behavior of a function refers to the behavior of a function as $x \to +\infty$ or $x \to -\infty$.
If $f(x) \to L$ as $x \to +\infty$ or $x \to -\infty$, L is the limit of f at (positive or negative) infinity.

Limits at Infinity (an informal view)

Let L be a real number.

1. The statement $\lim_{x\to\infty} f(x) = L$ means that we can make the value of $f(x)$ as close as we please to L by choosing a large enough value of x.
2. The statement $\lim_{x\to-\infty} f(x) = L$ means that we can make the value of $f(x)$ as close as we please to L by choosing a negative value of x with sufficiently large absolute value, i.e. by choosing a value of x far enough to the left.

Consider the function $f(x) = \dfrac{3x^2 - 2x + 1}{x^2 + 2x + 9}$.



From the graph, it appears that $f(x) \to 3$ as $x \to \infty$ and as $x \to -\infty$.
Use your calculator to estimate how far to the right x would have to be for $f(x)$ to be within 0.01 units of 3.

Use your calculator to estimate how far to the left x would have to be for $f(x)$ to be within 0.01 units of 3.

Definition of a Horizontal Asymptote

The line $y = L$ is a horizontal asymptote of the graph of f if
$\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$.

A function f may have as many as two different horizontal asymptotes,
one to the right and one to the left.

Example 1 $\lim_{x\to\infty}\left(2+\frac{3}{x}-\frac{5}{x^2}\right) =$

Example 2 $\lim_{x\to\infty}\frac{8x^2-3x+5}{2x^2+5x-7} =$

Example 3 $\lim_{x\to\infty}\frac{6x-1}{2x^2+5} =$

Limits of Rational Functions as $x \to +\infty$ or $x \to -\infty$ (Quick Method)

$$\lim_{x\to\infty}\frac{c_0+c_1x+c_2x^2+\ldots+c_{n-1}x^{n-1}+c_nx^n}{d_0+d_1x+d_2x^2+\ldots+d_{m-1}x^{m-1}+d_mx^m} = \lim_{x\to\infty}\frac{c_nx^n}{d_mx^m}$$

and

$$\lim_{x\to-\infty}\frac{c_0+c_1x+c_2x^2+\ldots+c_{n-1}x^{n-1}+c_nx^n}{d_0+d_1x+d_2x^2+\ldots+d_{m-1}x^{m-1}+d_mx^m} = \lim_{x\to-\infty}\frac{c_nx^n}{d_mx^m}$$

Example 4 Find each limit using the quick method.

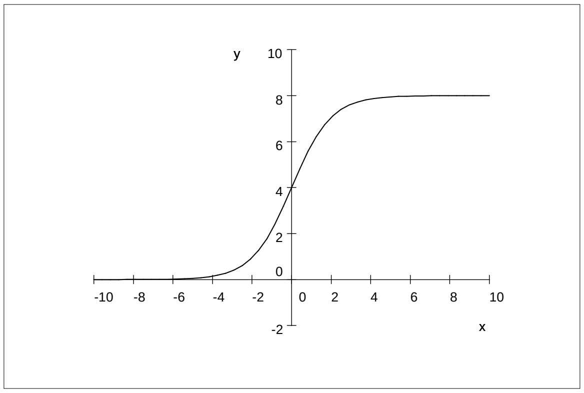
a. $\lim_{x\to\infty}\frac{8x-3}{2x^2+5}$

b. $\lim_{x\to\infty}\frac{8x-3}{2x+5}$

c. $\lim_{x\to-\infty}\frac{8x-3}{2x^2+5}$

d. $\lim_{x\to-\infty}\frac{8x-3}{2x+5}$

Example 5 Show that the logistic function $f(x) = \dfrac{8}{1+e^{-x}}$ has two different horizontal asymptotes.

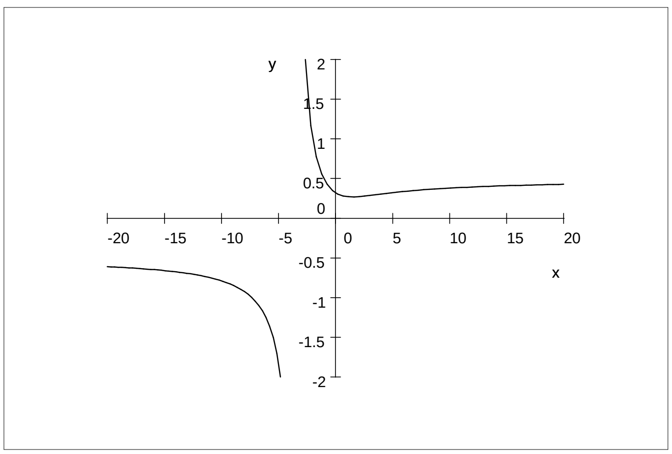


$$f(x) = \frac{8}{1+e^{-x}}$$

Limits at Infinity Involving Radicals

Example 6 Find $\lim\limits_{x\to\infty} \sqrt[3]{\dfrac{27x^2 - 3x + 5}{8x^2 + 5x - 1}}$.

Example 7 Find $\lim\limits_{x\to\infty} \dfrac{\sqrt{x^2+5}}{2x+7}$ and $\lim\limits_{x\to-\infty} \dfrac{\sqrt{x^2+5}}{2x+7}$.



$$f(x) = \frac{\sqrt{x^2+5}}{2x+7}$$

End Behavior of Trigonometric, Exponential, and Logarithmic Functions

Example 8 Evaluate each limit.

a. $\lim_{x\to\infty} \cos x$

e. $\lim_{x\to-\infty} e^x =$

Example 9 Evaluate each limit.

a. $\lim_{x\to\infty} \cos e^x$

b. $\lim_{x\to-\infty} \cos e^x =$

c. $\lim_{x\to\infty} e^{\cos x} =$

Infinite Limits at Infinity

1. The statement $\lim_{x\to\infty} f(x) = \infty$ means that for each positive number M, there is a corresponding number $N > 0$ such that $f(x) > M$ whenever $x > N$.

   The statement $\lim_{x\to\infty} f(x) = -\infty$ means that for each negative number M, there is a corresponding number $N > 0$ such that $f(x) < M$ whenever $x > N$.

2. The statement $\lim_{x\to-\infty} f(x) = \infty$ means that for each positive number M, there is a corresponding number $N < 0$ such that $f(x) > M$ whenever $x < N$.

   The statement $\lim_{x\to-\infty} f(x) = -\infty$ means that for each negative number M, there is a corresponding number $N < 0$ such that $f(x) < M$ whenever $x < N$.

Example 10 Find each limit.

a. $\lim_{x\to\infty}(-3x^5 + 200x^4 + 19x^2 - 15x + 3) =$

b. $\lim_{x\to-\infty}(5x^4 + 240x^3 + 22x^2 - 13) =$

c. $\lim_{x\to\infty}\dfrac{8x^2-3}{2x+5}$

d. $\lim_{x\to\infty}\dfrac{8x^3-3}{2x+5}$

e. $\lim_{x\to-\infty}\dfrac{8x^2-3}{2x+5}$

f. $\lim_{x\to-\infty}\dfrac{8x^3-3}{2x+5}$

g. $\lim_{x\to\infty}\ln x =$

h. $\lim_{x\to\infty}e^x =$

i. $\lim_{x\to\infty}\ln\left(\dfrac{1}{x}\right) =$