3.4 The Chain Rule

THEOREM (The Chain Rule). If g is differentiable at the point x and f is differentiable at the point $g(x)$, then the composition $f \circ g$ is differentiable at the point x.
Moreover, if $y = f(g(x))$ and $u = g(x)$, then $y = f(u)$ and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Also, note that $\frac{dy}{du} = \frac{d}{du}[y] = \frac{d}{du}[f(u)] = f'(u) = f'(g(x))$
and $\frac{du}{dx} = \frac{d}{dx}[u] = \frac{d}{dx}[g(x)] = g'(x)$.
So, the derivative of $f(g(x))$ can also be expressed as $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$.

Example 1 Find $\frac{dy}{dx}$ if $y = (8x^2 - 2x + 3)^4$.

The Generalized Derivative Formula

Since y is a function of u $(y = f(u))$, it is often helpful to abbreviate the Chain Rule as follows:

$$\frac{d}{dx}[f(u)] = f'(u)\frac{du}{dx}$$

Example 2 Find $\frac{dy}{dx}$ if $y = (3x^2 + 4x + 5)^3$.

Example 3 Find $\frac{dy}{dx}$ if $y = \frac{d}{dx}\left[\sqrt{x^5 + 10x}\right]$.

Example 4 Find $\frac{d}{dx}[\sec(x^3)]$

Example 5 Find $\frac{d}{dx}[\sec^3(x)]$

Example 6 Find $\frac{d}{dx}\left[\sin\left((x^2+5x+3)^3\right)\right]$

Example 7 Find $\frac{d}{dx}[\sin^3(x^2+5x+3)]$

Example 8 Find $f'(x)$ if $f(x) = x^2\sqrt{1-x^2}$

Example 9 Find $f'(x)$ if $f(x) = \frac{x}{\sqrt[3]{x^2+4}}$

The Derivative of the Natural Logarithm Function

Let be a differentiable function of x.

1. $\frac{d}{dx}[\ln x] = \frac{1}{x},\ x > 0$ 2. $\frac{d}{dx}[\ln u] = \frac{1}{u}\frac{du}{dx} = \frac{u'}{u},\ u > 0$

Remember the number e from Algebra. It is called the natural base e, and is often taken to be the limit $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ or $\lim\limits_{v\to\infty}\left(1+\frac{1}{v}\right)^v$, if you're not so crazy about the letter n.
We will use this limit to find $\frac{d}{dx}[\log_b x]$ where $\log_b x$ is the logarithm of x to an arbitrary base b.

$$\begin{aligned}
\frac{d}{dx}[\log_b x] &= \lim_{h\to0}\frac{\log_b(x+h)-\log_b x}{h}\\
&= \lim_{h\to0}\frac{\log_b \frac{x+h}{x}}{h}\\
&= \lim_{h\to0}\frac{1}{h}\log_b\frac{x+h}{x}\\
&= \lim_{h\to0}\frac{1}{h}\log_b\left(1+\frac{h}{x}\right)\\
&= \lim_{v\to\infty}\frac{v}{x}\log_b\left(1+\frac{1}{v}\right) && \left(\text{if we let } v=\frac{x}{h}\right)\\
&= \lim_{v\to\infty}\frac{1}{x}\,\frac{v}{1}\log_b\left(1+\frac{1}{v}\right)\\
&= \frac{1}{x}\lim_{v\to\infty}\left(\log_b\left(1+\frac{1}{v}\right)^v\right)\\
&= \frac{1}{x}\log_b\left(\lim_{v\to\infty}\left(1+\frac{1}{v}\right)^v\right) && (\text{because } \log_b x \text{ is continuous})\\
&= \frac{1}{x}\log_b(e) && \left(\text{because } \lim_{v\to\infty}\left(1+\frac{1}{v}\right)^v = e\right)\\
&= \frac{1}{x}\,\frac{\ln e}{\ln b} && (\text{by the Change-of-Base Formula})\\
&= \frac{1}{x\ln b}
\end{aligned}$$

The following list is a summary of the derivative formulas for logarithmic functions.

a. $\frac{d}{dx}[\log_b x] = \frac{1}{x\ln b}$ b. $\frac{d}{dx}[\ln x] = \frac{1}{x}$

c. $\frac{d}{dx}[\log_b u] = \frac{1}{u\ln b}\frac{du}{dx}$ d. $\frac{d}{dx}[\ln u] = \frac{1}{u}\frac{du}{dx}$

Example 10 Find each derivative.

a. $\frac{d}{dx}[\log_5(x)]$ b. $\frac{d}{dx}[\log_3(x^4-4x^2+5)]$ c. $\frac{d}{dx}[\ln(\sin x)]$

Example 11 Find $\frac{dy}{dx}$ if $y = \ln\left(\frac{x^5 \sqrt[3]{x^2+5}}{\tan^3 x}\right)$.

Example 12 Find each derivative.

a. Find $\frac{d}{dx}[\ln|x|]$.

b. Find $\frac{d}{dx}[\ln|\cos x|]$