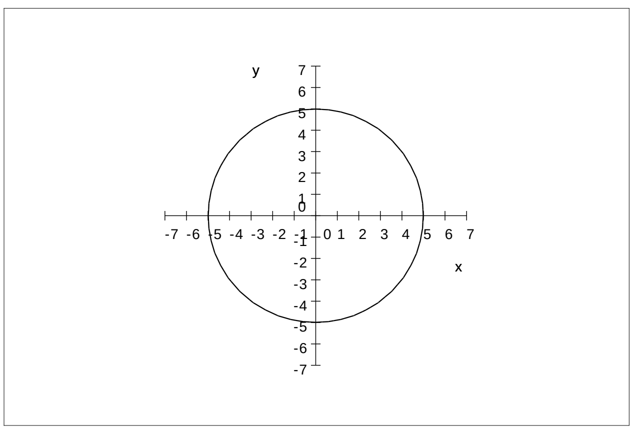
3.5 Implicit Differentiation

If we were to sketch the graph of the equation $2xy - x = 3y + 2$, we would see that the graph passes the vertical line test. I.e. the graph is that of a function, so the equation $2xy - x = 3y + 2$ determines a function.

Whenever possible, we will solve equations such as $2xy - x = 3y + 2$ for the variable y and express them *explicitly* as y in terms of x: $y = \dfrac{x+2}{2x-3}$

In some cases solving for y will not be possible or, simply, will not be convenient, so we will explore in this section how to find the derivative of a function such as $2xy - x = 3y + 2$ that *implicitly* defines y as a function of x.
Moreover, an equation whose graph does not pass the vertical line test will define more than one function of x, and the method of differentiation the we look at in this section will enable us to find $\dfrac{dy}{dx}$ nevertheless. The process of finding $\dfrac{dy}{dx}$ from an equation that *does not* express y explicitly as a function of x is called **implicit differentiation**.

Example 1 The graph of $x^2 + y^2 = 25$ is a circle centered at the origin with a radius of 5.
The upper half (semicircle) of this circle defines a function given by $y = \sqrt{25 - x^2}$. Also, the lower half (semicircle) of this circle defines a function given by $y = -\sqrt{25 - x^2}$.



Find the derivatives of $y = \sqrt{25 - x^2}$ and $y = -\sqrt{25 - x^2}$ explicitly and then implicitly.
Then find the value $\dfrac{dy}{dx}$ at the point $(3, 4)$ and at the point $(4, -3)$.

Example 2 Differentiate each expression as indicated. Assume that the variables x, y, and t may mutually depend on each other.

a. $\frac{d}{dx}[x^3] =$

b. $\frac{d}{dx}[y^3] =$

c. $\frac{d}{dt}[x^3] =$

d. $\frac{d}{dx}[\sin x] =$

e. $\frac{d}{dx}[\sin y] =$

f. $\frac{d}{dt}[\sin x] =$

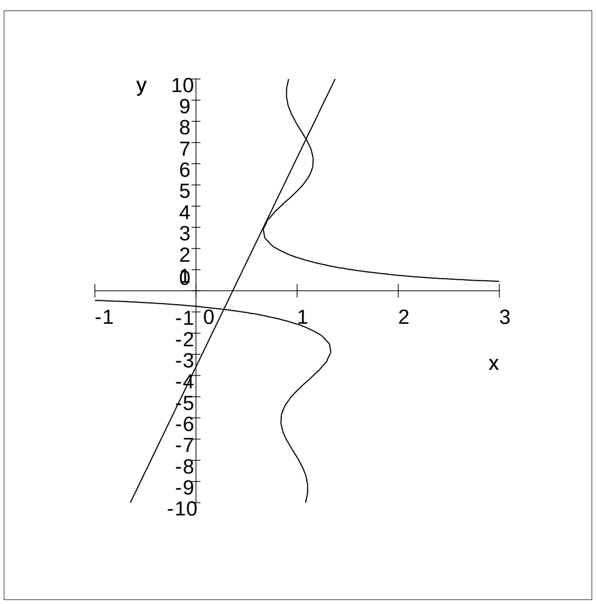
g. $\frac{d}{dx}[x^2y^2] =$

h. $\frac{d}{dy}[x^2y^2] =$

i. $\frac{d}{dt}[x^2y^2] =$

Example 3 Use implicit differentiation to find $\frac{dy}{dx}$ if $y + \cos y = xy$.

Find the slope of the tangent line to the graph at the point $\left(\frac{\pi - 1}{\pi}, \pi\right)$.



Example 4 Use implicit differentiation to find $\frac{dy}{dx}$ if $xy^2 + 3x^3y - y = 3$.

Example 5 a. Use implicit differentiation to find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ if $4x^2 + 9y^2 = 36$.

b. Find the slope of the tangent line to the graph at the points $(0,2)$ and $\left(1, \frac{4\sqrt{2}}{3}\right)$.

c. Find the rate at which the slope of the tangent line is changing with respect to x at the point $(0,2)$.

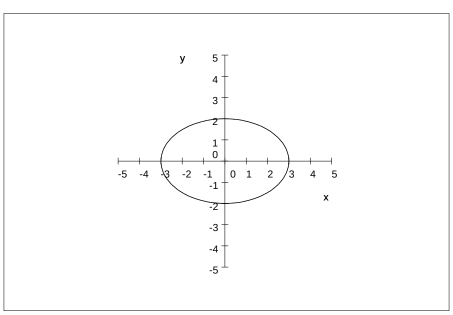
$$\frac{d}{dx}[4x^2] + \frac{d}{dx}[9y^2] = \frac{d}{dx}[36]$$

$$8x + 18y\frac{dy}{dx} = 0$$

$$18y\frac{dy}{dx} = -8x$$

$$\frac{dy}{dx} = \frac{-8x}{18y}$$

$$\boxed{\frac{dy}{dx} = \frac{-4x}{9y}}$$



$$\frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}\left[\frac{-4x}{9y}\right]$$

$$\frac{d^2y}{dx^2} = \frac{9y\frac{d}{dx}[-4x] - (-4x)\frac{d}{dx}[9y]}{(9y)^2}$$

$$\frac{d^2y}{dx^2} = \frac{9y(-4) - (-4x)\left(9\frac{dy}{dx}\right)}{(9y)^2}$$

$$\frac{d^2y}{dx^2} = \frac{-36y + 36x\frac{dy}{dx}}{81y^2}$$

$$\frac{d^2y}{dx^2} = \frac{-36y + 36x\left(\frac{-4x}{9y}\right)}{81y^2}$$

$$\frac{d^2y}{dx^2} = \frac{-36y - \frac{16x^2}{y}}{81y^2}$$

$$\frac{d^2y}{dx^2} = \frac{-36y^2 - 16x^2}{81y^3}$$

$$\frac{d^2y}{dx^2} = \frac{-4(4x^2 + 9y^2)}{81y^3}$$

$$\frac{d^2y}{dx^2} = \frac{-4(36)}{81y^3} = -\frac{16}{9y^3}$$

Example 6

Use logarithmic differentiation (whatever that is) to find $\frac{dy}{dx}$ if $y = \frac{x^2 \sin^4 x}{\sqrt[3]{4x - 9}}$.