4.8 Differentials

Tangent Line Approximation

If the point $(c, f(c))$ is a point of differentiability for the function f, the graphs of f and the tangent line to the graph of f at this point appear to be almost identical, provided that we zoom in very close on the point $(c, f(c))$.

Since the equation of the tangent line is linear, it is usually easier to evaluate than the original function f, so our initial purpose in this section is to use the linear function determined by the tangent line at c as a way of approximating the function values of f at points close to c.

Using the point-slope formula, we can see that the tangent line to the graph of $y = f(x)$ at the point $(c, f(c))$ has the equation $y - f(c) = f'(c)(x - c)$.

This equation can be rewritten as $y = f(c) + f'(c)(x - c)$.

Since the graphs of the function $y = f(x)$ and its tangent line $y = f(c) + f'(c)(x - c)$ are close together when x is sufficiently close to c, we can say that $f(x) \approx f(c) + f'(x)(x - c)$ for values of x that are fairly close to c.

The equation $y = f(c) + f'(c)(x - c)$ is the **tangent line approximation of** f **at** c.

Example 1 Find the tangent line approximation for $f(x) = \sqrt{x}$ at $c = 16$.
Use it to approximate $\sqrt{16.3}$, $\sqrt{17}$, and $\sqrt{20}$.

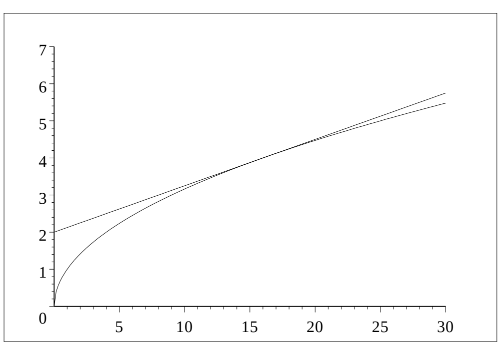
$f'(x) = \dfrac{1}{2\sqrt{x}}$, so $f'(16) = \dfrac{1}{2\sqrt{16}} = \dfrac{1}{8}$. Also $f(16) = \sqrt{16} = 4$.

$y = f(c) + f'(c)(x - c)$ then becomes $y = 4 + \dfrac{1}{8}(x - 16) = \dfrac{1}{8}x + 2$.

$\sqrt{16.1} \approx \dfrac{1}{8}(16.3) + 2 \approx 4.0375$ Compare this to the calculated value $\sqrt{16.3} \approx 4.0373258$.

$\sqrt{17} \approx \dfrac{1}{8}(17) + 2 \approx 4.125$ Compare this to the calculated value $\sqrt{17} \approx 4.1231056$.

$\sqrt{20} \approx \dfrac{1}{8}(20) + 2 \approx 4.5$ Compare this to the calculated value $\sqrt{20} \approx 4.4721359$.



Notice that as we move farther and farther away from $c = 16$ the approximations seem to agree less and less with the calculated values.

Differential Notation

We have seen that if $y = f(x)$ is a differentiable function, we can write $\frac{dy}{dx} = f'(x)$.

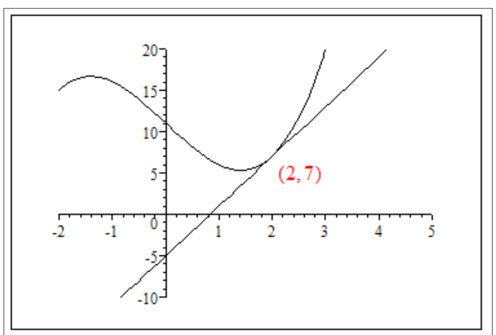
Up until this point, we have not thought of $\frac{dy}{dx}$ as a ratio, but rather as a special notation used to denote the derivative of y with respect to x. We can begin to think of $\frac{dy}{dx}$ as a ratio by making the following definitions:

First, we define the symbol dx to be an independent variable that can take on any value.
If f is differentiable at x, then we define dy by the formula $dy = f'(x)dx$.
This is called **differential notation**.

For any value of dx other than 0, we then have the equation $\frac{dy}{dx} = f'(x)$ where the left-hand side can actually be thought of as "dy divided by dx."

Example 2 Let $y = \sqrt{x}$. Find $\frac{dy}{dx}$ and write the equation $\frac{dy}{dx} = f'(x)$ in differential notation.
If $x = 16$ and $dx = \Delta x = 0.3$, find and compare Δy and dy.

Example 3 Use differentials and the graph of f to approximate the following.
(Round your answers to two decimal places.)



$f(x) = x^3 - 6x + 11$

a. $f(2.06) \approx$

b. $f(1.9) \approx$

Error Propagation in Applications

Suppose the quantities y and x are related by the function $y = f(x)$ and a value of y is to be determined by using a measured value of x and then evaluating the function f at x to find y.

Let x denote the measured value of x.
Let $x + \Delta x$ denote the true value of x.
Then Δx denotes the error which occurs in the measurement.
The expression $f(x + \Delta x)$ represents the true value of y which would result from evaluating f at $x + \Delta x$, and $f(x)$ denotes the value of y that is based on the measured value of x.
The difference $f(x + \Delta x) - f(x)$ is called the *propagated error* and is denoted by Δy.
$\Delta y = f(x + \Delta x) - f(x)$.

Our goal here is to express the propagated error Δy as an approximation in terms of the measurement error Δx.
In many cases, the differential dy will serve as a good approximation for Δy, and will be easier to compute. For this reason, when we want to find an approximate value for Δy, we will let $dx = \Delta x$ and compute dy using the differential notation $dy = f'(x)dx$.

Important! Δy and dy are usually different, but close in value, so we compute dy directly to estimate the value of Δy. On the other hand, Δx and dx *are always equal, and are simply different notations used to denote the same quantity.*
In the examples that follow, we will use the differential dx rather than Δx to denote the measurement error.

Example 4 A 12-foot ladder leaning against a wall makes an angle θ with the floor. If the top of the ladder is y feet up the wall, express y in terms of θ and then approximate the error in estimating y if $\theta = 60^o \pm 0.5^o$.

Example 5 The area of a circle is to computed from a measured value of its circumference. If the circumference is measured to be 48.7 cm $\pm\ 0.2$ cm,

a. Use differentials to estimate the error in the calculated area.

b. What is the percentage error in the measurement of the circumference?

c. What is the percentage error in the calculation of the area?

Calculating Differentials

When more than one function of an independent variable is present, the following differential formulas may be helpful.

Let u and v be differential functions of x. Then,

$$d[cu] = cdu$$
$$d[u \pm v] = du \pm dv$$
$$d[uv] = udv + vdu$$
$$d\left[\frac{u}{v}\right] = \frac{vdu - udv}{v^2}$$

Example 6 The circumference of a right circular cylinder is measured to be 95.7 cm with a possible error of ± 0.05 cm. The height of the cylinder is measured to be 275 cm with a possible error of ± 0.5 cm. Find the volume and the use differentials to approximate the propagated error in computing the volume of the cylinder.