3.3 The Product and Quotient Rules

THEOREM (The Product Rule). If f and g are differentiable at $x,$ then so is the product fg and

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$
$$= f(x)g'(x) + g(x)f'(x)$$

In words, *the derivative of a product of two functions is equal to the first times the derivative of the second plus the second times the derivative of the first*.

At this point Mr. Mizell will probably show you the proof because he thinks it's really cool. Just smile and nod and pretend to look interested. It is found on p. 140 of our textbook.

Example 1 Find $h'(x)$ if $h(x) = (3x-7)(2x^2+5)$ two ways.

a. Multiply it out and use the power rule.

b. Use the product rule.

Example 2 Find $f'(x)$ if $f(x) = (3x^4+4x)\sqrt{x}$

THEOREM (The Quotient Rule). If f and g are differentiable at x and $g(x) \neq 0$,

then $\frac{f}{g}$ is differentiable at x and $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$.

In words, *lo dee hi minus hi dee lo, over lo lo*.

Mr. Mizell will probably not bother you with the proof of the Quotient Rule. Although it is arguably even cooler than the proof of the Product Rule, it is similar in nature and is found on p. 142 of our textbook.

Example 3 Let $f(x) = \frac{2x-3}{x^2+1}$. Find $f'(x)$.

Example 4 Let $f(x) = \frac{3x^4+2x^2+5}{x^2}$. Find $f'(x)$ two ways.

a. Use the quotient rule.

b. Write the expression as the sum of three terms and use the power rule

Example 5 Find the derivative of each trigonometric function using the quotient rule.

a. $\frac{d}{dx}[\tan x] =$

b. $\frac{d}{dx}[\cot x] =$

c. $\frac{d}{dx}[\sec x] =$

d. $\frac{d}{dx}[\csc x] =$

Example 6 Find the derivative of each function.

a. $y = x \sin x$

b. $y = 2 \sin x \cos x$

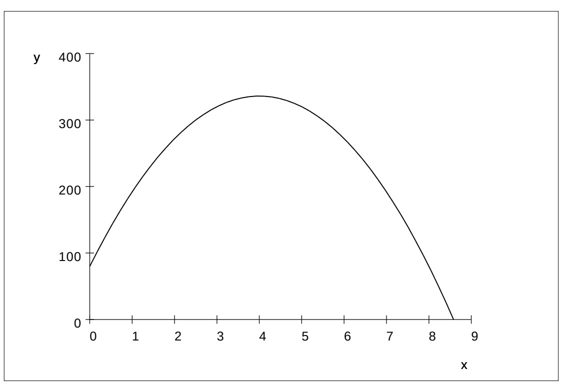
Higher Order Derivatives

If the derivative f' of a function f is differentiable, then we will denote $(f')'$ by f'', and f'' will be called the **second derivative** of f. If subsequent derivatives exist,
$(f'')' = f'''$ is called the **third derivative** of f.
$(f''')' = f^{(4)}$ is called the **fourth derivative** of f.
If $n \geq 4$, the n^{th} derivative of f will be denoted by $f^{(n)}$.

Other notation for derivatives:

Original function	y	$f(x)$	y	$f(x)$	y
First derivative	y'	$f'(x)$	$\frac{dy}{dx}$	$\frac{d}{dx}[f(x)]$	$D_x[y]$
Second derivative	y''	$f''(x)$	$\frac{d^2y}{dx^2}$	$\frac{d^2}{dx^2}[f(x)]$	$D_x^2[y]$
Third derivative	y'''	$f'''(x)$	$\frac{d^3y}{dx^3}$	$\frac{d^3}{dx^3}[f(x)]$	$D_x^3[y]$
Fourth derivative	$y^{(4)}$	$f^{(4)}(x)$	$\frac{d^4y}{dx^4}$	$\frac{d^4}{dx^4}[f(x)]$	$D_x^4[y]$
nth derivative	$y^{(n)}$	$f^{(n)}(x)$	$\frac{d^ny}{dx^n}$	$\frac{d^n}{dx^n}[f(x)]$	$D_x^n[y]$

Example 7 The height (in feet) of an arrow t seconds after being fired is given by $s(t) = -16t^2 + 128t + 80.$
Find and interpret the first derivative $s'(t)$ and the second derivative $s''(t)$.



Example 8 Let $f(x) = x^5 + x^4 + x^3 + x^2 + x + 1.$
Find the first six derivatives of f.