3.3 Evaluating Limits Analytically

THEOREM (Three Basic Limits).

$$\lim_{x\to a} b = b \qquad \lim_{x\to c} x = c \qquad \lim_{x\to c} x^n = c^n$$

Example 1

a. $\lim_{x\to 2} 5 =$ b. $\lim_{x\to 2} x =$ c. $\lim_{x\to x} x^3 =$

THEOREM. *If* $\lim_{x\to c} f(x) = L$ *and* $\lim_{x\to c} g(x) = K$, *then*

(a) $\lim_{x\to c}[f(x) + g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = L + K$

(b) $\lim_{x\to c}[f(x) - g(x)] = \lim_{x\to c} f(x) - \lim_{x\to c} g(x) = L - K$

(c) $\lim_{x\to c}[f(x)g(x)] = \lim_{x\to c} f(x) \lim_{x\to c} g(x) = LK$

(d) $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \dfrac{L}{K}$ if $K \neq 0$

(e) $\lim_{x\to c}[f(x)]^n = \left[\lim_{x\to c} f(x)\right]^n = L^n$

With a repeated application of part (a), it follows that $\lim kf(x) = k\lim f(x)$.

Example 1 Find $\lim_{x \to 2}(x^3 + 3x + 9)$ and justify each step.

THEOREM. For any polynomials $p(x)$ and $q(x)$ and any real number a,
$\lim_{x \to a} p(x) = p(a)$ and $\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$, whenever $a \neq 0$.

Example 2 Apply the previous theorem and evaluate each limit.

a. $\lim_{x \to 8}(x^2 + 4x + 5)$

b. $\lim_{x \to 3} \frac{x^2 + 2x + 1}{x - 1}$

Theorem Let n be a postive integer. The following limit is valid for all c if n is odd, and is valid for $c > 0$ if n is even.

$$\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$$

Theorem The Limit of a Composite Function
If f and g are functions such that $\lim_{x \to c} g(x) = L$ and $\lim_{x \to L} f(x) = f(L)$, then
$\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right) = f(L)$

Example 3 Let $f(x) = x^2 - 2x + 5$ and let $g(x) = \dfrac{3x-6}{\sqrt{4x-4}-2}$.

Find $\lim_{x \to 2} f(g(x))$.

THEOREM Limits of Transcendental Functions
Let c be a real number in the domain of the given trigonometric function.

1. $\lim_{x \to c} \sin x = \sin c$
2. $\lim_{x \to c} \cos x = \cos c$
3. $\lim_{x \to c} \tan x = \tan c$
4. $\lim_{x \to c} \cot x = \cot c$
5. $\lim_{x \to c} \sec x = \sec c$
6. $\lim_{x \to c} \csc x = \csc c$
7. $\lim_{x \to c} a^x = a^c$
8. $\lim_{x \to c} \ln x = \ln c$

Example 4

a. $\lim_{x \to 2} \sin \dfrac{\pi(x-1)}{4} =$

b. $\lim_{x \to \pi} (x \cos x) =$

b. $\lim_{x \to e^4} \sqrt{\ln x}$

Theorem Functions that Agree at All But One Point

Let c be a real number and let $f(x) = g(x)$ for all $x \neq c$ in an open interval containing c. If the limit of $g(x)$ as x approaches c exists, then the limit of $f(x)$ also exists and $\lim_{x \to c} f(x) = \lim_{x \to c} g(x)$.

Example 5 Find each limit.

a. $\lim_{x \to 3} \dfrac{x^2 - 3x + 2}{x - 2} =$

b. $\lim_{x \to 2} \dfrac{x^2 - 3x + 2}{x - 2} =$

$\lim_{x \to 1} \dfrac{7x^3 - 26x^2 + 29x - 10}{7x^3 + 16x^2 - 43x + 20} =$

Limits Involving (Square-root) Radicals (Rationalization Technique)

Example 6 Evaluate each limit without the use a calculator.

a. $\lim\limits_{x \to 9} \dfrac{3 - \sqrt{x}}{x - 9}$

b. $\lim\limits_{x \to 2} \dfrac{3x - 6}{\sqrt{4x - 4} - 2}$

THEOREM. (*The Squeeze Theorem*). Let f, g, and h be functions satisfying
$g(x) \leq f(x) \leq h(x)$
for all x in some open interval containing the point c, with the possible exception that the inequalities
need not hold at c. If g and h have the same limit as x approaches c, say
$\lim\limits_{x \to c} g(x) = \lim\limits_{x \to c} h(x) = L$
then f also has this limit as x approaches c, that is $\lim\limits_{x \to c} f(x) = L$.

THEOREM. (a) $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$ (b) $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x} = 0$

The proof of (a) is interesting and involves The Squeezing Theorem.
It is found on page 85 of your textbook.

Example 7 Find each limit.

(a) $\lim\limits_{x \to 0} \dfrac{\sin 3x}{2x}$ ‘ (b) $\lim\limits_{x \to 0} \dfrac{\tan 2x}{x}$

Example 8 Find each limit.

(a) $\lim_{\theta \to 0} \frac{\theta^2}{1 - \cos\theta}$

(b) $\lim_{x \to 0} \frac{1 - \cos 3x}{1 - \cos 2x}$

Example 9 Find each limit.

(a) $\lim_{x \to \frac{\pi}{6}} \frac{\frac{1}{2} - \sin x}{x - \frac{\pi}{6}}$

(b) $\lim_{x \to \frac{\pi}{6}} \frac{\frac{\sqrt{3}}{2} - \cos x}{\frac{\pi}{6} - x}$

Example 10 Use the Squeeze Theorem to find $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right)$.