4.1 Extrema on an Interval

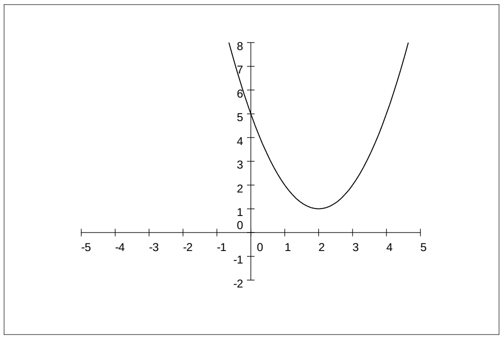
DEFINITION. A function f is said to have an **absolute maximum** on an interval I at the point x_0 if $f(x_0)$ is the largest value of f on I; that is, $f(x_0) \geq f(x)$ for all x in I. Similarly, f is said to have an **absolute minimum** on I at the point x_0 if $f(x_0)$ is the smallest value of f on I; that is, $f(x_0) \leq f(x)$ for all x in I. If f has either an absolute maximum or absolute minimum on I at x_0 then f is said to have an **absolute extremum** on I at x_0.

THEOREM. (Extreme-Value Theorem). If a function f is continuous on a finite closed interval $[a, b]$, then f has both an absolute maximum and an absolute minimum on $[a, b]$.

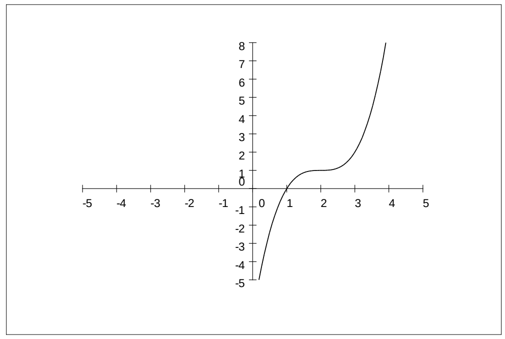
DEFINITION. A function f is said to have a **relative maximum** at x_0 if there is an open interval containing x_0 on which $f(x_0)$ is the largest value, that is, $f(x_0) \geq f(x)$ for all x in that interval. Similarly, f is said to have a **relative minimum** at x_0 if there is an open interval containing x_0 on which $f(x_0)$ is the smallest value, that is, $f(x_0) \leq f(x)$ for all x in that interval. If f has either a relative maximum or a relative minimum at x_0, then f is said to have a **relative extremum** at x_0.

THEOREM. If a function f has any relative extrema, then they occur either at numbers x where $f'(x) = 0$ or at numbers where f is not differentiable. These numbers are called the **critical numbers** of the function f.

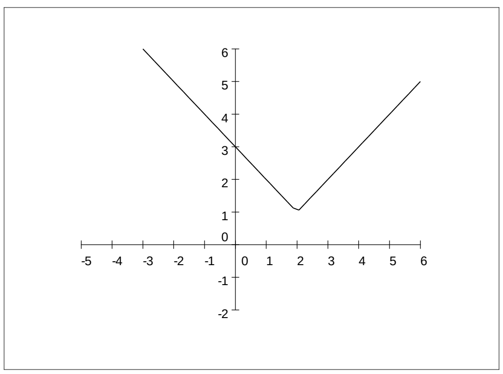
Example 1 Find the critical number of each function and indicate whether or not a relative extremum occurs there.



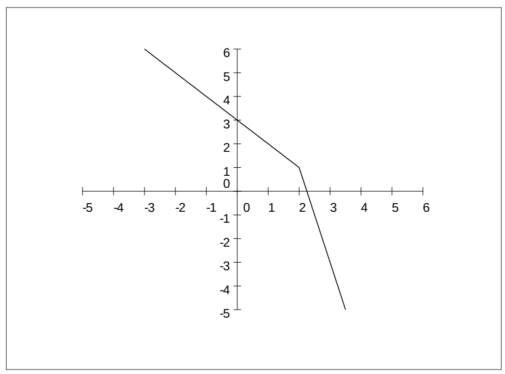
a. $f(x) = (x-2)^2 + 1$



b. $f(x) = (x-2)^3 + 1$



c. $f(x) = |x-2| + 1$



d. $f(x) = \begin{cases} -x+3 & if \quad x \le 2 \\ -4x+9 & if \quad x > 2 \end{cases}$

Finding Absolute Extrema on Finite Closed Intervals

THEOREM. If f is continuous on an open interval (a, b), and if f has an absolute extremum on this interval, then it must occur at a critical point of f.

The absolute extrema of a continuous function on a closed interval $[a, b]$ must occur either at endpoints of the interval or at critical points within the interval. This leads to the following three-step procedure for finding the absolute extrema of a function on a closed interval.

Step 1. Find the critical points of the function f in (a, b). Remember, these may be points at which $f'(x) = 0$ (stationary points) or points at which f is not differentiable.

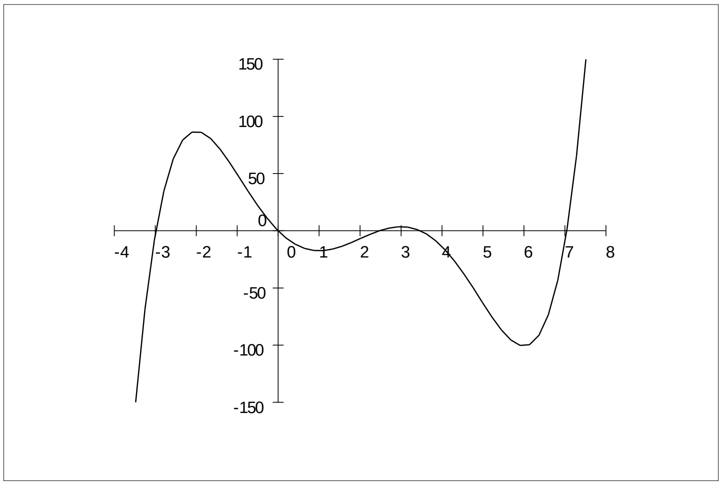
Step 2. Evaluate f at each of these critical points and at the endpoints of the interval.

Step 3. The largest of the values in Step 2 is the absolute maximum value of f on $[a, b]$ and the smallest value is the absolute minimum. Keep in mind that the absolute maximum or the absolute maximum may occur at more than one point within the interval $[a, b]$.

Example 2 Find the absolute maximum and absolute minimum of the function $f(x) = \frac{1}{5}x^5 - 2x^4 + \frac{7}{3}x^3 + 18x^2 - 36x$ on each of the intervals indicated below.

To save time $f'(x)$ has been computed and expressed in factored form.

$f'(x) = (x+2)(x-1)(x-3)(x-6)$



$f(x) = \frac{1}{5}x^5 - 2x^4 + \frac{7}{3}x^3 + 18x^2 - 36x$

a. [-3, 0]

b. [-3, 2]

c. [0, 4]

d. [0, 7]