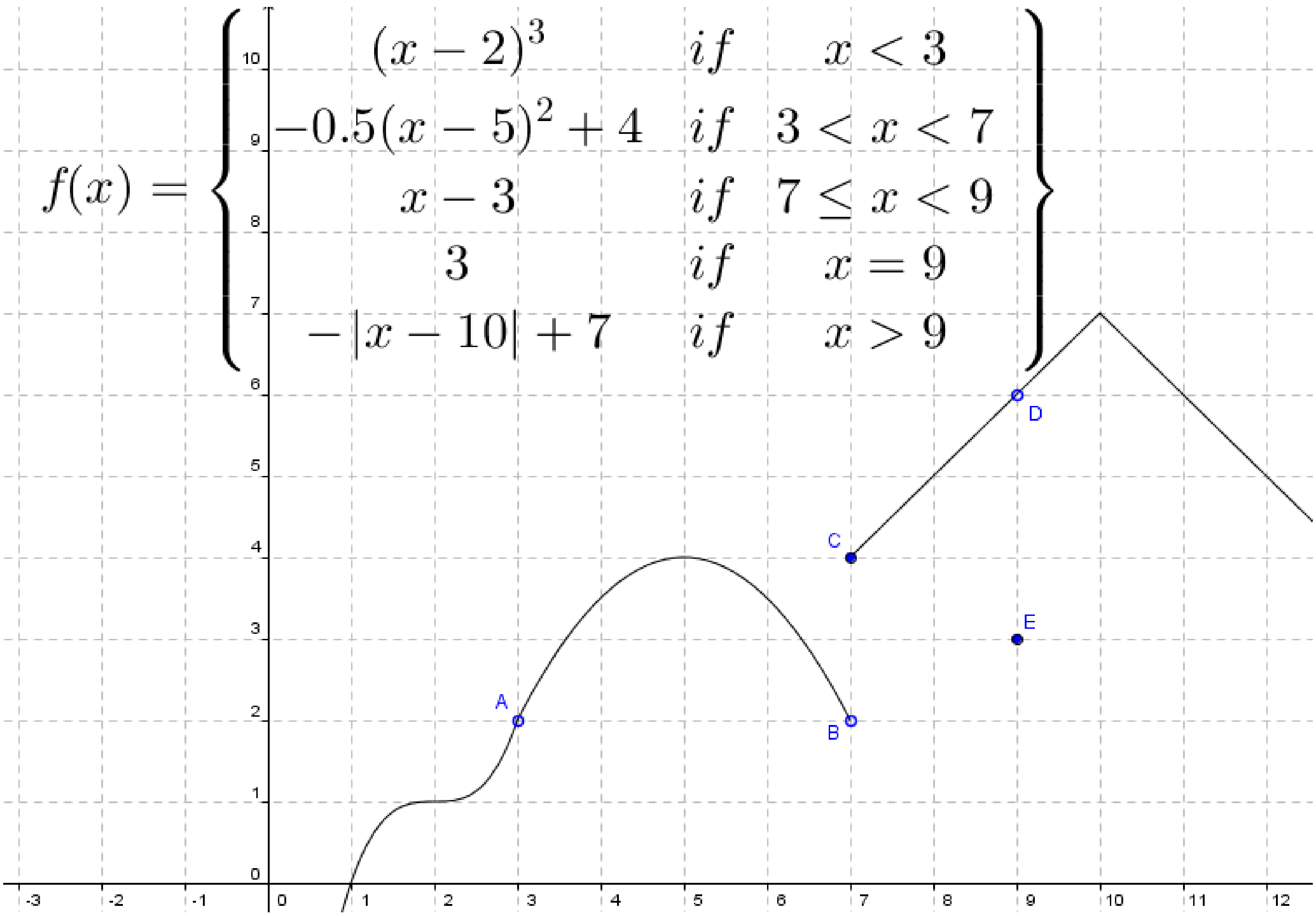
2.4 Continuity

Prior to studying calculus, we have considered a function to be continuous if its graph can be drawn on paper without having to "pick up" the writing utensil. Although this definition will continue to be largely effective, we can make the definition of continuity more precise using limits.

Continuity at a Point

If a function is defined in an open interval containing the number c, except possibly at c itself, there are three primary ways in which the function may fail to be continuous at c.



Definition of Continuity at a Point

A function f is said to be **continuous at** c if the following conditions are satisfied:

1. $f(c)$ is defined.
2. $\lim_{x \to c} f(x)$ exists.
3. $\lim_{x \to c} f(x) = f(c)$

A function is said to be **continuous on an open interval** (a, b) if it is continuous at every point in that interval. A function that is continuous on $(-\infty, \infty)$ is said to be **everywhere continuous.**

Example 1 Determine whether or not the function $f(x) = \dfrac{(x+4)(x-3)}{(x-3)}$ is continuous at $x = 3$ and if not why not.

Example 2 Determine whether or not the function $h(x) = \begin{cases} \dfrac{(x+4)(x-3)}{(x-3)} & \mathit{if} \quad x \neq 3 \\ 5 & \mathit{if} \quad x = 3 \end{cases}$

is continuous at $x = 3$ and if not why not.

Example 3 Define a function h which agrees with the function g (above) at every point except $x = 3$ so that it will be continuous at $x = 3$.

$$h(x) = \begin{cases} \dfrac{(x+4)(x-3)}{(x-3)} & \mathit{if} \quad x \neq 3 \\ & \mathit{if} \quad x = 3 \end{cases} \qquad \text{or simply} \quad h(x) =$$

The functions f and g in Example 1 and 2 above are said to have **removable discontinuities** at $x = 3$ because $\lim\limits_{x\to 3} f(x)$ and $\lim\limits_{x\to 3} g(x)$ exist. The discontinuities can be "removed" simply by defining or redefining the functions at the single domain value $x = 3$ to equal the value of the limit.

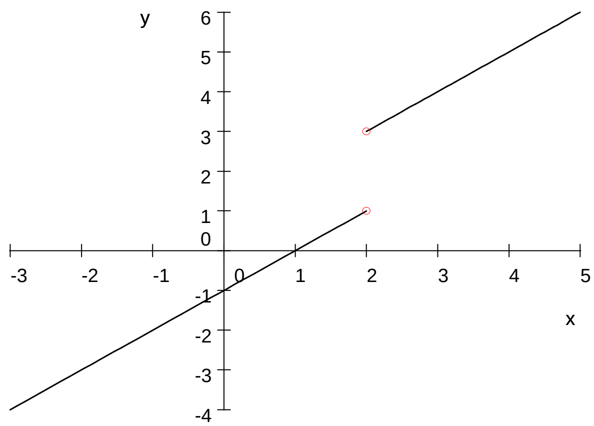
One-Sided Limits and Continuity on a Closed Interval

Let f be a function defined in an interval extending to the right of c.
$\lim\limits_{x\to c^+} f(x) = L$ is the limit of $f(x)$ as x approaches c from the right.
This one-sided limit L exists if for each $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $c - \delta < x < c$ then $|f(x) - L| < \varepsilon$.

Let f be a function defined in an interval extending to the left of c.
$\lim\limits_{x\to c^-} f(x) = L$ is the limit of $f(x)$ as x approaches c from the left.
This one-sided limit L exists if for each $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $c < x < c + \delta$ then $|f(x) - L| < \varepsilon$.

Example 4 Let $f(x) = \begin{cases} x+1 & if \quad x > 2 \\ x-1 & if \quad x < 2 \end{cases}$.

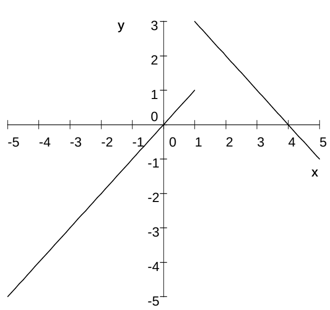
a. $\lim\limits_{x \to 2^-} f(x) =$ b. $\lim\limits_{x \to 2^+} f(x) =$ c. $\lim\limits_{x \to 2} f(x) =$

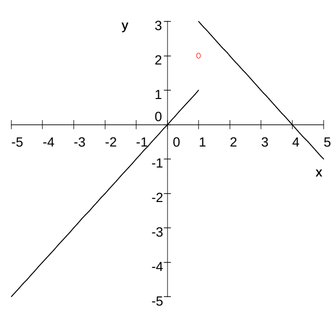


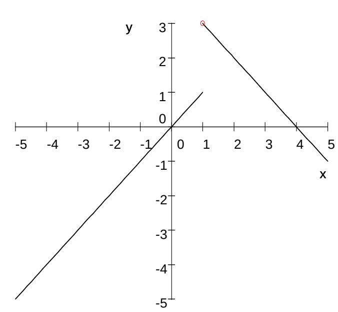
THE RELATIONSHIP BETWEEN ONE-SIDED AND TWO-SIDED LIMITS.
The two-sided limit of a function f exists at a point a if and only if the one-sided limits exist at that point and have the same value; that is,
$\lim\limits_{x \to a} f(x) = L$ if and only if $\lim\limits_{x \to a^+} f(x) = L$ and $\lim\limits_{x \to a^-} f(x) = L$.

Example 5 Find the limit of each function as x approaches 1 from the right and as x approaches 1 from the left.

a. $f(x) = \begin{cases} x & if \quad x < 1 \\ -x+4 & if \quad x > 1 \end{cases}$ b. $g(x) = \begin{cases} x & if \quad x < 1 \\ 2 & if \quad x = 1 \\ -x+4 & if \quad x > 1 \end{cases}$ c. $h(x) = \begin{cases} x & if \quad x < 1 \\ 3 & if \quad x = 1 \\ -x+4 & if \quad x > 1 \end{cases}$



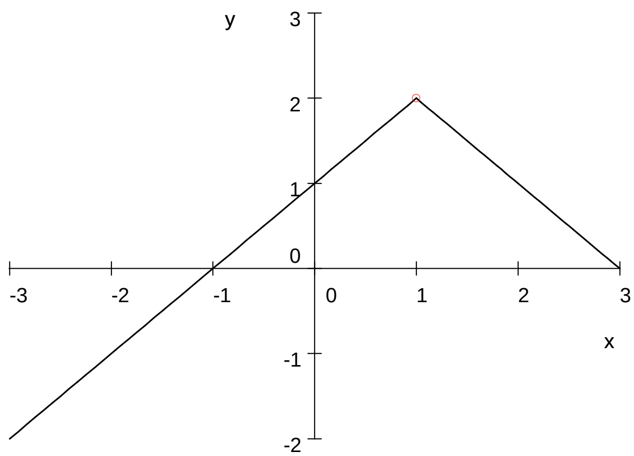




$\lim\limits_{x \to 1^+} f(x) =$ $\lim\limits_{x \to 1^+} g(x) =$ $\lim\limits_{x \to 1+} h(x) =$

$\lim\limits_{x \to 1^-} f(x) =$ $\lim\limits_{x \to 1^-} g(x) =$ $\lim\limits_{x \to 1^-} h(x) =$

Example 5 Let $f(x) = \begin{cases} x+1 & if \quad x < 1 \\ -x+3 & if \quad x > 1 \end{cases}$. Find $\lim_{x \to 1^-} f(x)$, $\lim_{x \to 1^+} f(x)$, and $\lim_{x \to 1} f(x)$.



Example 6 Let $f(x) = \begin{cases} x^2 - 1 & if \quad x < 0 \\ x - 2 & if \quad 0 < x < 4 \\ \sqrt{x} & if \quad x > 4 \end{cases}$. Find each limit.

a. $\lim_{x \to 0^-} f(x) =$

b. $\lim_{x \to 0^+} f(x) =$

c. $\lim_{x \to 0} f(x) =$

d. $\lim_{x \to 4^-} f(x) =$

e. $\lim_{x \to 4^+} f(x) =$

f. $\lim_{x \to 4} f(x) =$

Example 7 $\lim_{x \to 3^-} \sqrt{9 - x^2} =$

Definition of Continuity on a Closed Interval

A function f is said to be **continuous on the closed interval** $[a, b]$
if it is continuous on the open interval (a, b) and
$\lim_{x \to a^+} f(x) = f(a)$ and $\lim_{x \to b^-} f(x) = f(b)$.
If limits are satisfied we say f is **continuous from the right** at a and **continuous from the left** at b.

Example 8 Discuss the continuity of $f(x) = \sqrt{9 - x^2}$.

Example 9 Discuss the continuity of the function $f(x) = \begin{cases} 5 & if \quad x = -2 \\ x + 1 & if \quad -2 < x < 3 \\ 4 & if \quad x = 3 \end{cases}$.

Properties of Continuity

THEOREM
If the functions f and g are continuous at $x = c$, then
(a) bf is continuous at c for any real number b.
(b) $f + g$ and $f - g$ are continuous at c.
(c) fg is continuous at c.
(d) $\frac{f}{g}$ is continuous at c if $g(c) \neq 0$ and has a discontinuity at c if $g(c) = 0$.

The following types of functions are continuous at every point in their domains.
Polynomial functions.
Rational functions.
Radical functions.
Trigonometric functions.
Exponential and logarithmic functions.

Example 10 Indicate the intervals where each function is continuous.
Indicate whether any specific discontinuities are removable discontinuities, jump discontinuities, or vertical asymptotes.

a. $f(x) = \frac{x + 3}{x - 7}$ b. $g(x) = \frac{x^2 + x - 12}{x^2 - x - 20}$ c. $h(x) = \frac{\sqrt{100 - x^2}}{3 - \sqrt{x + 1}}$

Example 11 Find an equation relating a and b so that the following function will be continuous at $x = 1$.

$$f(x) = \begin{cases} x^2 + 5ax & \text{if } x \le 1 \\ ax + b & \text{if } x > 1 \end{cases}$$

THEOREM. *Let* $\lim$ *stand for one of the limits* $\lim\limits_{x\to c}$, $\lim\limits_{x\to c^+}$, $\lim\limits_{x\to c^-}$, $\lim\limits_{x\to+\infty}$, or $\lim\limits_{x\to-\infty}$.
If $\lim g(x) = L$ and the function f is continuous at L, then $\lim f(g(x)) = f(L)$.
That is, $\lim f(g(x)) = f(\lim g(x))$.

THEOREM.
(a) If the function g is continuous at c, and the function f is continuous at $g(c)$, then the composition $f \circ g$ is continuous at c.
(b) If the function g is continuous everywhere and the function f is continuous everywhere, then the composition $f \circ g$ is continuous everywhere.

Example 12 Discuss the continuity of each function.

a. $f(x) = \sin(2x - 3)$

b. $h(x) = |\cos x|$

THEOREM Intermediate Value Theorem
If f is continuous on the closed interval $[a,b]$ and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a,b]$ such that $f(c) = k$.

Example 13 What does the IVT tell you about zeros of $f(x) = x^3 - 6x^2 + 3x + 16$ when applied to each of the following intervals?

a. $[2,3]$

b. $[4,5]$

c. $[3,4]$

d. $[2,5]$