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Notes and Practice Problems for OpenStax Calculus I

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LIMIT OF A FUNCTION

$$\lim_{x\to a} f(x) = L \iff \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L.$$

If the left and right hand limits ***both exist but are not equal*** *or if* ***either of them does not exist****, then the two-sided limit does not exist.*

1. Let $f(x) = \begin{cases} 12 + x - x^2, & \text{if } x < 3 \\ x^2 - 3, & \text{if } x \geq 3 \end{cases}$. Find the following limits, if they exist.
   (a) $\lim_{x\to 3^-} f(x)$
   (b) $\lim_{x\to 3^+} f(x)$
   (c) $\lim_{x\to 3} f(x)$

2. Let $f(x) = \begin{cases} 7 - x^2, & \text{if } x < -1 \\ 4x + 10, & \text{if } -1 \leq x < 2 \\ -x^4 - 2, & \text{if } x \geq 2 \end{cases}$. Find the following limits, if they exist.
   (a) $\lim_{x\to -1} f(x)$
   (b) $\lim_{x\to 2} f(x)$

How to find $\lim_{x\to a} \frac{p(x)}{q(x)}$, where $\frac{p(x)}{q(x)}$ is a rational function?

- Use direct substitution if $q(a) \neq 0$.
- If $q(a) = 0$ but $p(a) \neq 0$, the function does not have a finite limit. In this case, the left and right hand limits are either $+\infty$ or $-\infty$, which can be determined by factoring $q(x)$ and observing the signs of $p(x)$ and the factors of $q(x)$. Note that the one-sided limits can be different: one $+\infty$ and the other $-\infty$.

$$\lim_{x\to a^-} \frac{p(x)}{q(x)} = \lim_{x\to a^+} \frac{p(x)}{q(x)} = +\infty \iff \lim_{x\to a} \frac{p(x)}{q(x)} = +\infty.$$

$$\lim_{x\to a^-} \frac{p(x)}{q(x)} = \lim_{x\to a^+} \frac{p(x)}{q(x)} = -\infty \iff \lim_{x\to a} \frac{p(x)}{q(x)} = -\infty.$$

- If $p(a) = q(a) = 0$, factor $p(x)$ and/or $q(x)$ and cancel common factors. Then you will have one the above two cases.

Evaluate the following limits, if they exist.

1. $\lim_{x\to 2} \frac{x^2 - 3x}{x + 2}$
2. $\lim_{x\to 2} \frac{x^2 - 2x}{x^2 - 4x + 2}$
3. (a) $\lim_{x\to 2^-} \frac{4}{(x-2)^3}$
   (b) $\lim_{x\to 2^+} \frac{4}{(x-2)^3}$
   (c) $\lim_{x\to 2} \frac{4}{(x-2)^3}$
4. (a) $\lim_{x\to -3^-} \frac{x^2 + 1}{(x+3)^8(x-2)}$

(b) $\lim_{x\to -3^+} \frac{x^2+1}{(x+3)^8(x-2)}$

(c) $\lim_{x\to -3} \frac{x^2+1}{(x+3)^8(x-2)}$

5. $\lim_{x\to 2} \frac{x^2-2x}{x^2-5x+6}$

6. $\lim_{x\to -3} \frac{x^2-9}{x^2+5x+6}$

7. $\lim_{x\to 0} \frac{3x}{4-(-2+x)^2}$

8. $\lim_{x\to 0} \left(\frac{1}{x} - \frac{5}{x(x-5)}\right)$

9. $\lim_{x\to 3} \left(\frac{1}{x-3} - \frac{4}{x^2-2x-3}\right)$

10. $\lim_{x\to -3} \frac{\frac{1}{x+2}+1}{x+3}$

11. $\lim_{x\to 0} \frac{x^2}{x-5+\frac{25}{x+5}}$

Special Trigonometic Limits

- $\lim_{x\to 0} \frac{\sin x}{x} = 1, \qquad \lim_{x\to 0} \frac{x}{\sin x} = 1$
- $\lim_{x\to 0} \frac{\tan x}{x} = 1, \qquad \lim_{x\to 0} \frac{x}{\tan x} = 1$
- $\lim_{x\to 0} \frac{1-\cos x}{x} = 0, \qquad \lim_{x\to 0} \frac{\cos x - 1}{x} = 0$

Find the following limits, if they exist.

1. $\lim_{x\to 0} \frac{\sin(5x)}{x}$

2. $\lim_{\theta\to 0} \frac{8\theta}{\tan(4\theta)}$

3. $\lim_{x\to 0} \left(\frac{\tan(x^2)}{x} + \frac{x^2}{\sin x}\right)$

4. $\lim_{t\to 0} \frac{\tan(2t)}{\sin(5t)}$

5. $\lim_{x\to 0} \frac{1-\cos(5x)}{10x}$

6. $\lim_{x\to 0} \frac{-4x+\cos(2x)-1}{3x}$

Reciprocal Trigonometric Indentities

$\csc x = \frac{1}{\sin x}, \qquad \sec x = \frac{1}{\cos x}, \qquad \cot x = \frac{1}{\tan x}$

Find the following limits, if they exist.

1. $\lim_{x\to 0} 2x\csc(4x)$

2. $\lim_{\theta\to 0} \frac{8\cos(2\theta)}{\theta\cot^2(4\theta)}$

3. $\lim_{x\to 0} \sin(2x)\cot(3x)$

4. $\lim_{t\to 0} \frac{\csc(2t)}{\cot(5t)}$

Find the limits, if they exist. Also, indicate if a limit is $+\infty$ or $-\infty$.

1. (a) $\lim_{x\to \frac{7\pi}{2}^-} \tan x$

   (b) $\lim_{x\to \frac{7\pi}{2}^+} \tan x$

   (c) $\lim_{x\to \frac{7\pi}{2}} \tan x$

2. (a) $\lim_{x\to -3\pi^-} \csc x$

   (b) $\lim_{x\to -3\pi^+} \csc x$

   (c) $\lim_{x\to -3\pi} \csc x$

CONTINUITY

Continuity at a point: A function f is said to be continuous at $x = a$ if

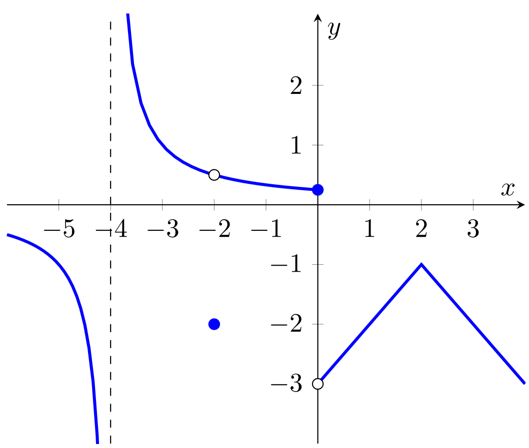
$$\lim_{x \to a} f(x) = f(a).$$

In other words, f is continuous at a if ALL three of the following conditions are true:

1. $f(a)$ is defined. (i.e., a is in the domain of f.)
2. $\lim_{x \to a} f(x)$ exists. (i.e., both one-sided limits at a exist and are equal.)
3. $\lim_{x \to a} f(x) = f(a)$.

If any of the above conditions is false, then we say that f is discontinuous at a, or that f has a discontinuity at a.

The continuity of f at a point simply means that the graph of f is unbroken at that point.



The function given by the graph above is discontinuous at -4, -2, and 0, but is continuous everywhere else. The function has an **infinite discontinuity** at -4, a **removable discontinuity** at -2 and a **jump discontinuity** at 0.

TYPES OF DISCONTINUITIES

Removable Discontinuity: f has a removable discontinuity at a if $\lim_{x \to a} f(x)$ exists, but $f(a)$ is either undefined or is not equal to the limit. In this case the graph of the function has a **hole** at $x = a$. This discontinuity can be 'repaired' by just filling in a single point in the graph (i.e., by defining or redefining $f(a)$ to be equal to the limit). No other type of discontinuity can be removed even by changing billions of function values.

Jump Discontinuity: f has a jump discontinuity at a if both one-sided limits of f at a exist, but are unequal.

Infinite Discontinuity: f has an infinite discontinuity at a if either one-sided limit of f at a is $+\infty$ or $-\infty$.

Continuity from the left: A function f is said to be continuous from the left at $x = a$ if

$$\lim_{x \to a^-} f(x) = f(a).$$

Continuity from the right: A function f is said to be continuous from the right at $x = a$ if

$$\lim_{x \to a^+} f(x) = f(a).$$

The function given above is left-continuous at 0, but not right-continuous.

Thus the *continuity at a point* means *continuity from the left as well as from the right* at that point.

Continuity on an open interval: A function f is said to be continuous on an open interval (a, b) if it is continuous at each and every point in that interval.

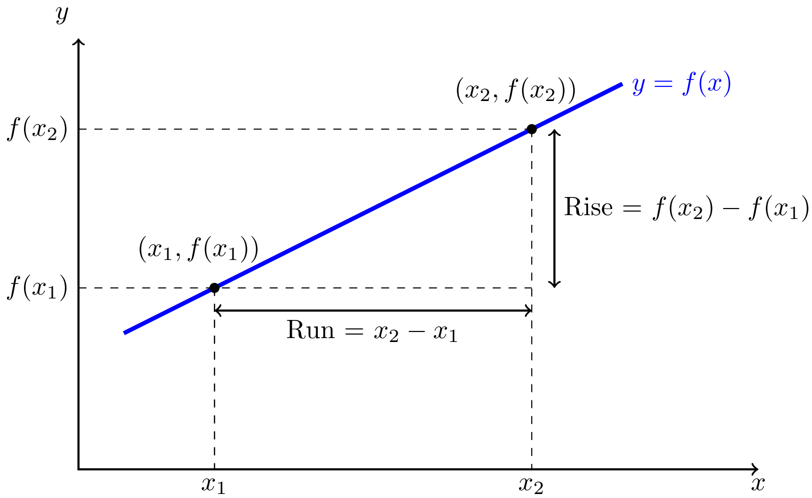
Continuity on a closed interval: A function f is said to be continuous on a closed interval $[a, b]$ if it is (i) continuous from the right at a, (ii) continuous from the left at b, and (iii) continuous at each point between a and b.

The function given by the graph above is continuous on $(-\infty, -4), (-4, -2), (-2, 0]$, and on $(0, \infty)$.

If f is continuous at every point in its domain, we simply say that f is a **continuous function.**
The Intermediate Value Theorem: If a function f is **continuous** on a **closed interval** $[a, b]$ and the outputs $f(a)$ and $f(b)$ have **opposite signs**, then there is at least one zero of f between a and b, i.e., there is at least one point c in (a, b) such that $f(c) = 0$.

Slope and Rate of Change

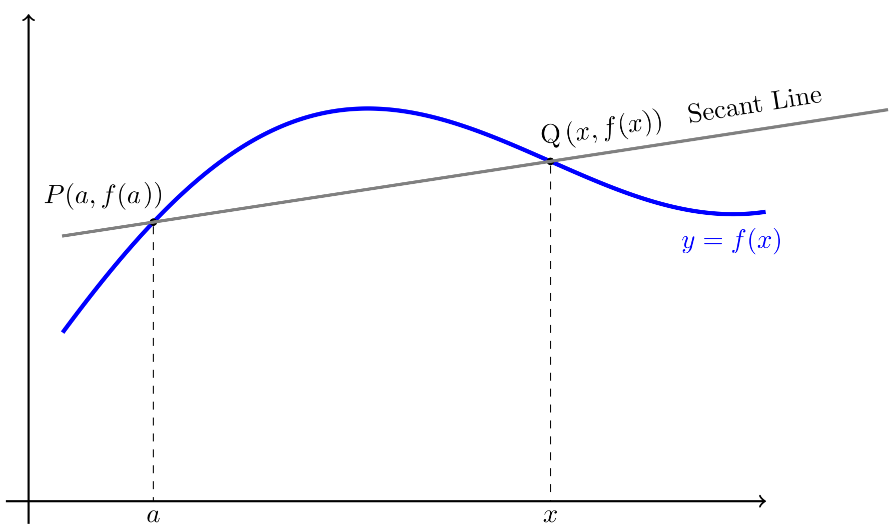
The graph of $f(x) = mx + b$ (or, $y = mx + b$) is a straight line with slope m. The slope m measures how fast the line rises or falls as we move from left to right along the line.



$$\boxed{\text{Slope } (m) = \frac{\text{Rise}}{\text{Run}} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.}$$

Thus, the slope m is the ratio of the *change in output* to the *change in input*, i.e., the **average rate of change** of $f(x)$ with respect to x over the interval $[x_1, x_2]$. Since the linear function changes at the same rate at all points, the slope m is also the **instantaneous rate of change** of $f(x)$ at *each* x.

Now let us consider a function whose graph is not a straight line. Instead of x_1 and x_2, we will use inputs a and x (You will see later that we will fix a and use x as a variable).



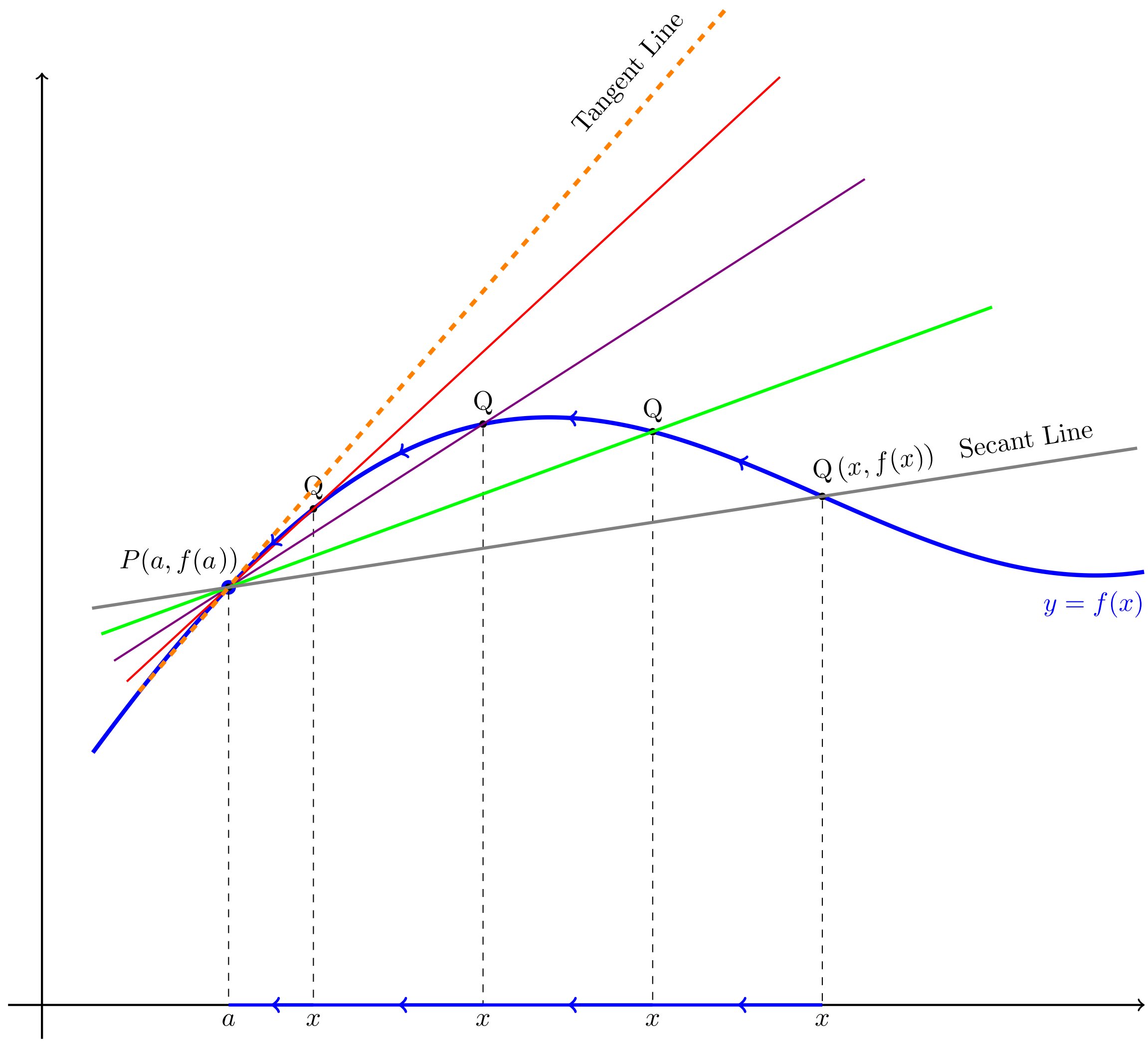
The **average rate of change** of f as the input changes from a to x is given by:

$$\frac{\text{change in output}}{\text{change in input}} = \frac{f(x) - f(a)}{x - a},$$

which is the **slope of the secant line** passing through $P(a, f(a))$ and $Q(x, f(x))$.

Is it also the instantaneous rate of change of f at each input as in the linear case? Of course not! The graph clearly shows that the function changes at different rates at different points. Then what is the **instantaneous rate of change** of $f(x)$ when the input is, for instance, a? In other words, what is the **slope of the curve** $y = f(x)$ at $P(a, f(a))$? Here the difficulty to compute the slope of the curve at P is that we do not have two points. We overcome this situation as follows:

We move the point $Q(x, f(x))$ along the curve closer and closer to $P(a, f(a))$. As $Q \to P$ along the curve (i.e., if $x \to a$), the slope $\dfrac{f(x) - f(a)}{x - a}$ of the secant line through P and Q approaches what we call *the slope of the curve at P*. In other words, the slope of the curve at P is the limit of $\dfrac{f(x) - f(a)}{x - a}$ as $x \to a$.



DEFINITIONS:

1. The **slope of the curve** $y = f(x)$ at $P(a, f(a))$ is

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

provided the limit exists.

2. The slope of the curve $y = f(x)$ at $P(a, f(a))$, if it exists, is called the **derivative of f at a**, and is denoted by $f'(a)$ (read as "f prime of a").

$$\boxed{f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad \text{(provided the limit exists).}}$$

3. f is said to be **differentiable** at a if $f'(a)$ exists, i.e., if f has a derivative at a.
4. The **tangent line** to the curve $y = f(x)$ at $P(a, f(a))$ is the straight line through P with slope $f'(a)$, if the derivative exists. The equation of the tangent line is:

$$\boxed{y - f(a) = f'(a)(x - a) \quad \text{or} \quad y = f(a) + f'(a)(x - a)}$$

If $\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = +\infty$ or $-\infty$, the function f is not differentiable at a, but for geometric reasons, we say that the curve $y = f(x)$ has a **vertical tangent** at $P(a, f(a))$, whose equation is: $x = a$.

If $f'(a)$ does not exist and $\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \neq \pm\infty$, the curve $y = f(x)$ has **no tangent line** at $P(a, f(a))$.

Alternative Formula for the Derivative:

In the definition, $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$, let us substitute $x - a = h$. Then $x = a + h$, and "$x \to a$" is equivalent to "$x - a \to a - a$", i.e.,"$h \to 0$". So the definition of the derivative takes the following form:

$$\boxed{f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \quad \text{(provided the limit exists).}}$$

Derivative as a Function: The above definition can be used to find the derivative of f at each point in the domain of f where it is differentiable. So, we can replace the fixed input a by a variable x to define f' as a function of x:

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \quad \text{(provided the limit exists).}}$$

Thus, f' is a function whose value at x is given by the limit of the **difference quotient** $\frac{f(x + h) - f(x)}{h}$ as h approaches 0. The notation y' is also used to denote the derivative of $y = f(x)$ with respect to x. The process of finding the derivative of a function is called **differentiation**.

The Leibniz Notation for Derivatives: In addition to the "prime notation", there are some other common notations for derivative. The most popular of these is the **Leibniz notation**. In Leibniz notation, the derivative of $y = f(x)$ is denoted by

$$\frac{dy}{dx}, \quad \frac{df}{dx}, \quad \text{or} \quad \frac{d}{dx}f(x),$$

where the sympol $\frac{d}{dx}$ indicates the *operation of differentiation* with respect to x. To denote $f'(a)$, the value of the derivative of $y = f(x)$ at $x = a$, we write:

$$\left.\frac{dy}{dx}\right|_{x=a}, \quad \left.\frac{df}{dx}\right|_{x=a}, \quad \text{or} \quad \left.\frac{d}{dx}f(x)\right|_{x=a}.$$

Note that if $y = g(t)$ is a function of input variable t, then its derivative with respect to t is denoted by

$$\frac{dy}{dt}, \quad \frac{dg}{dt}, \quad \text{or} \quad \frac{d}{dt}g(t).$$

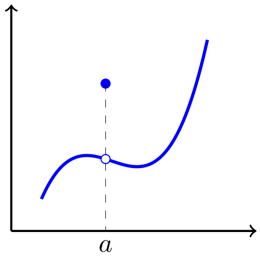
$D(f)(x)$ and $D_x(f)$ are some other alternative notations for the derivative of $y = f(x)$.

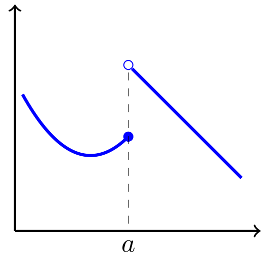
When does the derivatie of $f(x)$ NOT exist at $x = a$?

In simple terms, the differentiability of $f(x)$ at $x = a$ means that the graph of $f(x)$ is *continuous* (i.e., unbroken) and *smooth* (i.e., does not change direction abruptly) at $x = a$, and has a *non-vertical tangent line* at that point.

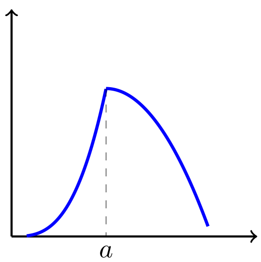
Three cases when $f'(a)$ does not exist:

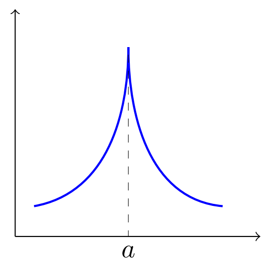
1. **Discontinuity at $x = a$.**



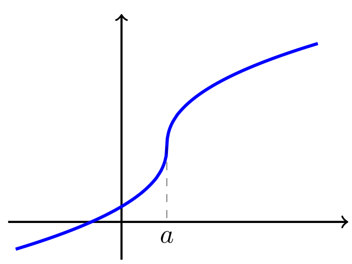


2. **A corner or a cusp at $x = a$.**





3. **Vertical tangent at $x = a$.**



Differentiation Rules:

Assume f, g, u and v are differentiable functions of x, and c is a constant.

Constant Rule: $\dfrac{d}{dx}(c) = 0$

Constant Multiple Rule: $(cu)' = cu'$

Sum Rule: $(u + v)' = u' + v'$

Difference Rule: $(u - v)' = u' - v'$

Product Rule: $(uv)' = u'v + uv'$

Quotient Rule: $\left(\dfrac{u}{v}\right)' = \dfrac{u'v - uv'}{v^2}$

Chain Rule: $\dfrac{d}{dx}\left(f(g(x)\right) = f'(g(x)) \cdot g'(x)$

Power Rule: $\dfrac{d}{dx}(x^n) = nx^{n-1}$

Derivatives of Trigonometric Functions:

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\sec x) = \sec x \, \tan x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\csc x) = -\csc x \, \cot x$$

Derivatives of Exponential Functions:

$$\frac{d}{dx}(e^x) = e^x \qquad \frac{d}{dx}(b^x) = (\ln b)\, b^x$$

Derivatives of Logarithmic Functions:

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \qquad \frac{d}{dx}(\log_b x) = \frac{1}{(\ln b)\, x}$$

1. Compute the following derivatives using the table below.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
0	-2	0	2	1/2
1	-3	2	0	1
2	1	4	-1	2
-1	0	-2	-2	2
-2	3	-3	-1	1

(a) Find $h'(2)$ if $h(x) = 5 - 5f(x)$.

(b) Find $k'(2)$ if $k(x) = x^4 + 2g(x)$.

(c) Find $p'(-2)$ if $p(x) = f(x)g(x)$.

(d) Find $F'(1)$ if $F(x) = \dfrac{f(x)}{g(x)}$.

(e) Find $G'(0)$ if $G(x) = f(g(x))$.

Find the derivatives.

2. (a) $\dfrac{d}{dx}(6 + 6x)$

(b) $\dfrac{d}{dx}(x^6 + 6^x + 6^6)$

(c) $\dfrac{d}{dx}(5e + 5e^x + 5^x)$

(d) $\dfrac{d}{dt}(\ln t + \log_5 t - \log t)$

(e) $\dfrac{d}{dx}(2\sin x - 3\cos x)$

(f) $\dfrac{d}{du}(\tan u + 5\sec u)$

(g) $\dfrac{d}{ds}(\cot s - \csc s)$

3. (a) Find $f'(-2)$ if $f(x) = 3x^2 - 9x + 5$.

(b) Find the slope of the graph of $g(x) = 5\cos x - \sin x + 8$ at $x = \pi/2$.

(c) Find the slope of the tangent line to the curve $y = 7x + e^x$ at $x = 0$.

(d) Find the instantaneous rate of change of $h(t) = 3 - 4\ln t$ when $t = 2$.

4. (a) $\dfrac{d}{dx}\left(x^5 + x^{-5} - x^{1/5} + x^{-1/5}\right)$

(b) $\dfrac{d}{dx}\left(1 + \dfrac{1}{x} + \dfrac{1}{x^2}\right)$

(c) $\dfrac{d}{dx}\left(\sqrt{x} + \dfrac{1}{\sqrt{x}}\right)$

(d) $\dfrac{d}{dx}\left(\dfrac{x}{6} - \dfrac{6}{x}\right)$

(e) $\dfrac{d}{dt}\left(\dfrac{2}{5}x^5 - 5x^{-\frac{2}{5}} - \dfrac{6}{\sqrt[3]{x}}\right)$

5. (a) $\dfrac{d}{dx}(x)^7$

(b) $\dfrac{d}{dx}\left(x^2 - 5x + 11\right)^7$

(c) $\dfrac{d}{dx}(\sin^7 x)$

6. (a) $\dfrac{d}{dx}(\sqrt{x})$

(b) $\dfrac{d}{dx}\left(\sqrt{x^4 + 3x^2 + 1}\right)$

(c) $\dfrac{d}{dx}(\sqrt{\cos x})$

7. (a) $\dfrac{d}{dx}\left(\dfrac{1}{x}\right)$

(b) $\dfrac{d}{dx}\left(\dfrac{1}{2x^5 - 1}\right)$

(c) $\dfrac{d}{dx}\left(\dfrac{1}{\ln x}\right)$

8. (a) $\dfrac{d}{dx}\left(\sin x - \sin(5x) + \sin(x^5)\right)$

(b) $\dfrac{d}{dx}(\cot x + 3\cot(\sqrt{x}) - 5\cot(\cos x))$

9. (a) $\dfrac{d}{dx}\left(e^x - e^{x^2} + e^{\cot x}\right)$

(b) $\dfrac{d}{dx}\left(2^x + 2^{x^2 - \sec x}\right)$

10. (a) $\dfrac{d}{dx}\left(\ln x + \ln(x^4 + 14) - 3\ln(\tan x + 3)\right)$

(b) $\dfrac{d}{dx}(\log_2 x + \log_2(1/x) - \log_2(\sqrt{x}))$

11. (a) $\dfrac{d}{dx}\left(\tan^3 x + \tan(x^3)\right)$

(b) $\dfrac{d}{dx}\left(\sqrt{\sin x} + \sin(\sqrt{x})\right)$

(c) $\dfrac{d}{dx}\left(\dfrac{1}{\ln x} - \ln\left(\dfrac{1}{x}\right)\right)$

(d) $\dfrac{d}{dx}(e^{\tan x} + \tan(e^x))$

(e) $\dfrac{d}{dx}(\ln(\sin x) + \sin(\ln x))$

12. (a) $\dfrac{d}{dx}(x^2 e^x)$

(b) $\dfrac{d}{dx}(x\ln x)$

(c) $\dfrac{d}{dx}\left((x^3 - 2x + 7)\sin x\right)$

(d) $\dfrac{d}{dx}\left((x^3 - 2x + 7)(x^2 - 7x + 11)\right)$

(e) $\dfrac{d}{dx}(\cot x\ \csc x)$

(f) $\dfrac{d}{dx}(2^x \log_2 x)$

13. (a) $\frac{d}{dx}\left(\frac{\sin x}{e^x}\right)$

(b) $\frac{d}{dx}\left(\frac{x}{\ln x}\right)$

(c) $\frac{d}{dx}\left(\frac{x^3}{x^2+4}\right)$

14. (a) $\frac{d}{dx}\left(\frac{x^2 \sin x}{e^x}\right)$

(b) $\frac{d}{dx}\left(\frac{\cos x}{x \ln x}\right)$

15. (a) $\frac{d}{dx}\left(2x^2(3x^4 - 9x + 7)\right)$

(b) $\frac{d}{dx}\left(\sqrt{x}\left(3x - 2\sqrt{x} + \frac{5}{\sqrt{x}}\right)\right)$

(c) $\frac{d}{dx}\left(\frac{2x^3 - 3x^2 + x - 2}{5x^2}\right)$

(d) $\frac{d}{dx}\left(\frac{x^3 - 5x + 3}{\sqrt{x}}\right)$

(e) $\frac{d}{dx}(\cos x \csc x)$

(f) $\frac{d}{dx}\left(\frac{1}{\cot x} + \frac{4\cos x}{\sin x}\right)$

(g) $\frac{d}{dx}\ln(x^{10})$

(h) $\frac{d}{dx}\left(e^{2\ln x}\right)$

(i) $\frac{d}{dx}\left(5^{3\log_5 x}\right)$

16. (a) Find $f'(x)$ if $f(x) = \sin x \ln x - \sin(\ln x)$.

(b) Find $g'(0)$ if $g(t) = e^{\tan t} - e^t \tan t$.