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Notes and Practice Problems for OpenStax Calculus I

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LIMIT OF A FUNCTION

\[ \lim_{x \to a} f(x) = L \iff \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L. \]

If the left and right hand limits both exist but are not equal or if either of them does not exist, then the two-sided limit does not exist.

1. Let \( f(x) = \begin{cases} 12 + x - x^2, & \text{if } x < 3 \\ x^2 - 3, & \text{if } x \geq 3 \end{cases} \) Find the following limits, if they exist.
   (a) \( \lim_{x \to 3^-} f(x) \)
   (b) \( \lim_{x \to 3^+} f(x) \)
   (c) \( \lim_{x \to 3} f(x) \)

2. Let \( f(x) = \begin{cases} 7 - x^2, & \text{if } x < -1 \\ 4x + 10, & \text{if } -1 \leq x < 2 \\ -x^4 - 2, & \text{if } x \geq 2 \end{cases} \) Find the following limits, if they exist.
   (a) \( \lim_{x \to -1} f(x) \)
   (b) \( \lim_{x \to 2} f(x) \)

How to find \( \lim_{x \to a} \frac{p(x)}{q(x)} \), where \( p(x) \) is a rational function?

- Use direct substitution if \( q(a) \neq 0 \).
- If \( q(a) = 0 \) but \( p(a) \neq 0 \), the function does not have a finite limit. In this case, the left and right hand limits are either \(+\infty\) or \(-\infty\), which can be determined by factoring \( q(x) \) and observing the signs of \( p(x) \) and the factors of \( q(x) \). Note that the one-sided limits can be different: one \(+\infty\) and the other \(-\infty\).

\[ \lim_{x \to a^-} \frac{p(x)}{q(x)} = \lim_{x \to a^+} \frac{p(x)}{q(x)} = +\infty \iff \lim_{x \to a} \frac{p(x)}{q(x)} = +\infty. \]

\[ \lim_{x \to a^-} \frac{p(x)}{q(x)} = \lim_{x \to a^+} \frac{p(x)}{q(x)} = -\infty \iff \lim_{x \to a} \frac{p(x)}{q(x)} = -\infty. \]

- If \( p(a) = q(a) = 0 \), factor \( p(x) \) and/or \( q(x) \) and cancel common factors. Then you will have one of the above two cases.

Evaluate the following limits, if they exist.

1. \( \lim_{x \to 2} \frac{x^2 - 3x}{x + 2} \)
2. \( \lim_{x \to 2} \frac{x^2 - 2x}{x^2 - 4x + 2} \)
3. \( \lim_{x \to -2} \frac{4}{(x - 2)^3} \)
4. \( \lim_{x \to 3} \frac{x^2 + 1}{(x + 3)^8(x - 2)} \)
Special Trigonometric Limits

- \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \), \( \lim_{x \to 0} \frac{x}{\sin x} = 1 \)
- \( \lim_{x \to 0} \frac{\tan x}{x} = 1 \), \( \lim_{x \to 0} \frac{x}{\tan x} = 1 \)
- \( \lim_{x \to 0} \frac{1 - \cos x}{x} = 0 \), \( \lim_{x \to 0} \frac{\cos x - 1}{x} = 0 \)

Find the following limits, if they exist.

1. \( \lim_{x \to 0} \frac{\sin(5x)}{x} \)
2. \( \lim_{\theta \to 0} \frac{8\theta}{\tan(4\theta)} \)
3. \( \lim_{x \to 0} \left( \frac{\tan(x^2)}{x} + \frac{x^2}{\sin x} \right) \)
4. \( \lim_{t \to 0} \frac{\tan(2t)}{\sin(5t)} \)
5. \( \lim_{x \to 0} \frac{1 - \cos(5x)}{10x} \)
6. \( \lim_{x \to 0} \frac{-4x + \cos(2x) - 1}{3x} \)

Reciprocal Trigonometric Identities

- \( \csc x = \frac{1}{\sin x} \), \( \sec x = \frac{1}{\cos x} \), \( \cot x = \frac{1}{\tan x} \)

Find the following limits, if they exist.

1. \( \lim_{x \to 0} 2x \csc(4x) \)
2. \( \lim_{\theta \to 0} \frac{8\cos(2\theta)}{\theta \cot^2(4\theta)} \)
3. \( \lim_{x \to 0} \sin(2x) \cot(3x) \)
4. \( \lim_{t \to 0} \frac{\csc(2t)}{\cot(5t)} \)

Find the limits, if they exist. Also, indicate if a limit is \( +\infty \) or \( -\infty \).

1. (a) \( \lim_{x \to \frac{\pi}{2}^-} \tan x \)
   (b) \( \lim_{x \to \frac{\pi}{2}^+} \tan x \)
   (c) \( \lim_{x \to \frac{\pi}{2}^+} \tan x \)
2. (a) \( \lim_{x \to -3\pi^-} \csc x \)
   (b) \( \lim_{x \to -3\pi^+} \csc x \)
   (c) \( \lim_{x \to -3\pi^+} \csc x \)
CONTINUITY

Continuity at a point: A function \( f \) is said to be continuous at \( x = a \) if
\[
\lim_{x \to a} f(x) = f(a).
\]

In other words, \( f \) is continuous at \( a \) if ALL three of the following conditions are true:

1. \( f(a) \) is defined. (i.e., \( a \) is in the domain of \( f \).)
2. \( \lim_{x \to a} f(x) \) exists. (i.e., both one-sided limits at \( a \) exist and are equal.)
3. \( \lim_{x \to a} f(x) = f(a) \).

If any of the above conditions is false, then we say that \( f \) is discontinuous at \( a \), or that \( f \) has a discontinuity at \( a \).

The continuity of \( f \) at a point simply means that the graph of \( f \) is unbroken at that point.

The function given by the graph above is discontinuous at \(-4\), \(-2\), and 0, but is continuous everywhere else. The function has an infinite discontinuity at \(-4\), a removable discontinuity at \(-2\) and a jump discontinuity at 0.

TYPES OF DISCONTINUITIES

Removable Discontinuity: \( f \) has a removable discontinuity at \( a \) if \( \lim_{x \to a} f(x) \) exists, but \( f(a) \) is either undefined or is not equal to the limit. In this case the graph of the function has a hole at \( x = a \). This discontinuity can be ‘repaired’ by just filling in a single point in the graph (i.e., by defining or redefining \( f(a) \) to be equal to the limit). No other type of discontinuity can be removed even by changing billions of function values.

Jump Discontinuity: \( f \) has a jump discontinuity at \( a \) if both one-sided limits of \( f \) at \( a \) exist, but are unequal.

Infinite Discontinuity: \( f \) has an infinite discontinuity at \( a \) if either one-sided limit of \( f \) at \( a \) is \( +\infty \) or \( -\infty \).

Continuity from the left: A function \( f \) is said to be continuous from the left at \( x = a \) if
\[
\lim_{x \to a^-} f(x) = f(a).
\]

Continuity from the right: A function \( f \) is said to be continuous from the right at \( x = a \) if
\[
\lim_{x \to a^+} f(x) = f(a).
\]
The function given above is left-continuous at 0, but not right-continuous.

Thus the continuity at a point means continuity from the left as well as from the right at that point.

Continuity on an open interval: A function $f$ is said to be continuous on an open interval $(a, b)$ if it is continuous at each and every point in that interval.

Continuity on a closed interval: A function $f$ is said to be continuous on a closed interval $[a, b]$ if it is (i) continuous from the right at $a$, (ii) continuous from the left at $b$, and (iii) continuous at each point between $a$ and $b$.

The function given by the graph above is continuous on $(-\infty, -4), (-4, -2), (-2, 0]$, and on $(0, \infty)$.

If $f$ is continuous at every point in its domain, we simply say that $f$ is a continuous function.

The Intermediate Value Theorem: If a function $f$ is continuous on a closed interval $[a, b]$ and the outputs $f(a)$ and $f(b)$ have opposite signs, then there is at least one zero of $f$ between $a$ and $b$, i.e., there is at least one point $c$ in $(a, b)$ such that $f(c) = 0$. 
Slope and Rate of Change

The graph of \( f(x) = mx + b \) (or, \( y = mx + b \)) is a straight line with slope \( m \). The slope \( m \) measures how fast the line rises or falls as we move from left to right along the line.

\[
\text{Slope (} m \text{)} = \frac{\text{Rise}}{\text{Run}} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.
\]

Thus, the slope \( m \) is the ratio of the change in output to the change in input, i.e., the average rate of change of \( f(x) \) with respect to \( x \) over the interval \([x_1, x_2]\). Since the linear function changes at the same rate at all points, the slope \( m \) is also the instantaneous rate of change of \( f(x) \) at each \( x \).

Now let us consider a function whose graph is not a straight line. Instead of \( x_1 \) and \( x_2 \), we will use inputs \( a \) and \( x \) (You will see later that we will fix \( a \) and use \( x \) as a variable).

The average rate of change of \( f \) as the input changes from \( a \) to \( x \) is given by:

\[
\frac{\text{change in output}}{\text{change in input}} = \frac{f(x) - f(a)}{x - a},
\]

which is the slope of the secant line passing through \( P(a, f(a)) \) and \( Q(x, f(x)) \).
Is it also the instantaneous rate of change of \( f \) at each input as in the linear case? Of course not! The graph clearly shows that the function changes at different rates at different points. Then what is the **instantaneous rate of change** of \( f(x) \) when the input is, for instance, \( a \)? In other words, what is the **slope of the curve** \( y = f(x) \) at \( P(a, f(a)) \)? Here the difficulty to compute the slope of the curve at \( P \) is that we do not have two points. We overcome this situation as follows:

We move the point \( Q(x, f(x)) \) along the curve closer and closer to \( P(a, f(a)) \). As \( Q \to P \) along the curve (i.e., if \( x \to a \)), the slope \( \frac{f(x) - f(a)}{x - a} \) of the secant line through \( P \) and \( Q \) approaches what we call the **slope of the curve at** \( P \). In other words, the slope of the curve at \( P \) is the limit of \( \frac{f(x) - f(a)}{x - a} \) as \( x \to a \).

**DEFINITIONS:**

1. The **slope of the curve** \( y = f(x) \) at \( P(a, f(a)) \) is
   \[
   \lim_{x \to a} \frac{f(x) - f(a)}{x - a},
   \]
   provided the limit exists.

2. The slope of the curve \( y = f(x) \) at \( P(a, f(a)) \), if it exists, is called the **derivative of** \( f \) **at** \( a \), and is denoted by \( f'(a) \) (read as “\( f \) prime of \( a \)”).
3. If \( f' \) exists at \( a \), then \( f \) is said to be **differentiable** at \( a \).

4. The **tangent line** to the curve \( y = f(x) \) at \( P(a,f(a)) \) is the straight line through \( P \) with slope \( f'(a) \), if the derivative exists. The equation of the tangent line is:

\[
y - f(a) = f'(a)(x - a) \quad \text{or} \quad y = f(a) + f'(a)(x - a)
\]

If \( \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = +\infty \) or \(-\infty\), the function \( f \) is not differentiable at \( a \), but for geometric reasons, we say that the curve \( y = f(x) \) has a vertical tangent at \( P(a,f(a)) \), whose equation is: \( x = a \).

If \( f'(a) \) does not exist and \( \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \neq \pm\infty \), the curve \( y = f(x) \) has no tangent line at \( P(a,f(a)) \).

**Alternative Formula for the Derivative:**

In the definition, \( f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \), let us substitute \( x - a = h \). Then \( x = a + h \), and “\( x \to a \)” is equivalent to “\( x - a \to a - a \)”, i.e., “\( h \to 0 \)”. So the definition of the derivative takes the following form:

\[
f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \quad \text{(provided the limit exists)}.
\]

**Derivative as a Function:** The above definition can be used to find the derivative of \( f \) at each point in the domain of \( f \) where it is differentiable. So, we can replace the fixed input \( a \) by a variable \( x \) to define \( f' \) as a function of \( x \):

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \quad \text{(provided the limit exists)}.
\]

Thus, \( f' \) is a function whose value at \( x \) is given by the limit of the difference quotient \( \frac{f(x + h) - f(x)}{h} \) as \( h \) approaches 0. The notation \( y' \) is also used to denote the derivative of \( y = f(x) \) with respect to \( x \). The process of finding the derivative of a function is called **differentiation**.

**The Leibniz Notation for Derivatives:** In addition to the “prime notation”, there are some other common notations for derivative. The most popular of these is the **Leibniz notation**. In Leibniz notation, the derivative of \( y = f(x) \) is denoted by

\[
\frac{dy}{dx}, \quad \frac{df}{dx}, \quad \text{or} \quad \frac{d}{dx} f(x),
\]

where the symbol \( \frac{d}{dx} \) indicates the operation of differentiation with respect to \( x \). To denote \( f'(a) \), the value of the derivative of \( y = f(x) \) at \( x = a \), we write:

\[
\left. \frac{dy}{dx} \right|_{x=a}, \quad \left. \frac{df}{dx} \right|_{x=a}, \quad \text{or} \quad \left. \frac{d}{dx} f(x) \right|_{x=a}.
\]

Note that if \( y = g(t) \) is a function of input variable \( t \), then its derivative with respect to \( t \) is denoted by

\[
\frac{dy}{dt}, \quad \frac{dg}{dt}, \quad \text{or} \quad \frac{d}{dt} g(t).
\]

\( D(f)(x) \) and \( D_x(f) \) are some other alternative notations for the derivative of \( y = f(x) \).
When does the derivative of $f(x)$ NOT exist at $x = a$?

In simple terms, the differentiability of $f(x)$ at $x = a$ means that the graph of $f(x)$ is continuous (i.e., unbroken) and smooth (i.e., does not change direction abruptly) at $x = a$, and has a non-vertical tangent line at that point.

Three cases when $f'(a)$ does not exist:

1. Discontinuity at $x = a$.

2. A corner or a cusp at $x = a$.

3. Vertical tangent at $x = a$. 
Differentiation Rules:

Assume \( f, g, u \) and \( v \) are differentiable functions of \( x \), and \( c \) is a constant.

**Constant Rule:**
\[
\frac{d}{dx}(c) = 0
\]

**Constant Multiple Rule:**
\[
(cu)' = cu'
\]

**Sum Rule:**
\[
(u + v)' = u' + v'
\]

**Difference Rule:**
\[
(u - v)' = u' - v'
\]

**Product Rule:**
\[
(uv)' = u'v + uv'
\]

**Quotient Rule:**
\[
\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}
\]

**Chain Rule:**
\[
\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)
\]

**Power Rule:**
\[
\frac{d}{dx}(x^n) = nx^{n-1}
\]

**Derivatives of Trigonometric Functions:**
\[
\begin{align*}
\frac{d}{dx}(\sin x) &= \cos x \\
\frac{d}{dx}(\sec x) &= \sec x \tan x \\
\frac{d}{dx}(\cos x) &= -\sin x \\
\frac{d}{dx}(\cot x) &= -\csc^2 x \\
\frac{d}{dx}(\tan x) &= \sec^2 x \\
\frac{d}{dx}(\csc x) &= -\csc x \cot x
\end{align*}
\]

**Derivatives of Exponential Functions:**
\[
\begin{align*}
\frac{d}{dx}(e^x) &= e^x \\
\frac{d}{dx}(b^x) &= (\ln b) b^x
\end{align*}
\]

**Derivatives of Logarithmic Functions:**
\[
\begin{align*}
\frac{d}{dx}(\ln x) &= \frac{1}{x} \\
\frac{d}{dx}(\log_b x) &= \frac{1}{(\ln b) x}
\end{align*}
\]
1. Compute the following derivatives using the table below.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
<th>$g(x)$</th>
<th>$f'(x)$</th>
<th>$g'(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-2</td>
<td>0</td>
<td>2</td>
<td>1/2</td>
</tr>
<tr>
<td>1</td>
<td>-3</td>
<td>2</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>4</td>
<td>-1</td>
<td>2</td>
</tr>
<tr>
<td>-2</td>
<td>0</td>
<td>-2</td>
<td>-2</td>
<td>2</td>
</tr>
</tbody>
</table>

(a) Find $h'(2)$ if $h(x) = 5 - 5f(x)$.
(b) Find $k'(2)$ if $k(x) = x^4 + 2g(x)$.
(c) Find $p'(-2)$ if $p(x) = f(x)g(x)$.
(d) Find $F'(1)$ if $F(x) = \frac{f(x)}{g(x)}$.
(e) Find $G'(0)$ if $G(x) = f(g(x))$.

Find the derivatives.

2. (a) $\frac{d}{dx}(6 + 6x)$
(b) $\frac{d}{dx}(x^6 + 6^x + 6^x)$
(c) $\frac{d}{dx}(5e + 5e^x + 5e^x)$
(d) $\frac{d}{dt}(\ln t + \log_3 t - \log t)$
(e) $\frac{d}{dx}(2 \sin x - 3 \cos x)$
(f) $\frac{d}{du}(\tan u + 5 \sec u)$
(g) $\frac{d}{ds}(\cot s - \csc s)$

3. (a) Find $f'(-2)$ if $f(x) = 3x^2 - 9x + 5$.
(b) Find the slope of the graph of $g(x) = 5\cos x - \sin x + 8$ at $x = \pi/2$.
(c) Find the slope of the tangent line to the curve $y = 7x + e^x$ at $x = 0$.
(d) Find the instantaneous rate of change of $h(t) = 3 - 4 \ln t$ when $t = 2$.

4. (a) $\frac{d}{dx}(x^5 + x^{-5} - x^{1/5} + x^{-1/5})$
(b) $\frac{d}{dx}\left(1 + \frac{1}{x} + \frac{1}{x^2}\right)$
(c) $\frac{d}{dx}\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)$
(d) $\frac{d}{dx}\left(\frac{x}{6} - \frac{6}{x}\right)$
(e) $\frac{d}{dt}\left(\frac{2}{5}x^5 - 5x^{2/5} - 6\sqrt{x}\right)$

5. (a) $\frac{d}{dx}(x^7)$
(b) $\frac{d}{dx}(x^2 - 5x + 11)^7$
(c) $\frac{d}{dx}(\sin^7 x)$

6. (a) $\frac{d}{dx}(\sqrt{x})$
(b) $\frac{d}{dx}(\sqrt{x^4 + 3x^2 + 1})$
(c) $\frac{d}{dx}(\sqrt{\cos x})$

7. (a) $\frac{d}{dx}\left(\frac{1}{x}\right)$
(b) $\frac{d}{dx}\left(\frac{1}{2x^3 - 1}\right)$
(c) $\frac{d}{dx}\left(\frac{1}{\ln x}\right)$

8. (a) $\frac{d}{dx}(\sin x - \sin(5x) + \sin(x^5))$
(b) $\frac{d}{dx}(\cot x + 3 \cot(\sqrt{x}) - 5 \cot(\cos x))$

9. (a) $\frac{d}{dx}(e^x - e^x + e^{\cot x})$
(b) $\frac{d}{dx}(2x + 2x^2 - \sec x)$

10. (a) $\frac{d}{dx}(\ln x + \ln(x^4 + 14) - \ln(\tan x + 3))$
(b) $\frac{d}{dx}(\log_2 x + \log_2(1/x) - \log_2(\sqrt{x}))$

11. (a) $\frac{d}{dx}(\tan^3 x + \tan(x^3))$
(b) $\frac{d}{dx}(\sqrt{\sin x + \sin(\sqrt{x})})$
(c) $\frac{d}{dx}\left(\frac{1}{\ln x} - \ln \left(\frac{1}{x}\right)\right)$
(d) $\frac{d}{dx}(e^{\tan x} + \tan(e^x))$
(e) $\frac{d}{dx}(\ln(\sin x) + \sin(\ln x))$

12. (a) $\frac{d}{dx}(x^2 e^x)$
(b) $\frac{d}{dx}(x \ln x)$
(c) $\frac{d}{dx}\left(\frac{(x^3 - 2x + 7) \sin x}{x^2}\right)$
(d) $\frac{d}{dx}\left(\frac{(x^3 - 2x + 7)(x^2 - 7x + 11)}{x^2}\right)$
(e) $\frac{d}{dx}(\cot x \csc x)$
(f) $\frac{d}{dx}(2^x \log_2 x)$
13. (a) \( \frac{d}{dx} \left( \frac{\sin x}{e^x} \right) \)
(b) \( \frac{d}{dx} \left( \frac{x}{\ln x} \right) \)
(c) \( \frac{d}{dx} \left( \frac{x^3}{x^2 + 4} \right) \)

14. (a) \( \frac{d}{dx} \left( \frac{x^2 \sin x}{e^x} \right) \)
(b) \( \frac{d}{dx} \left( \frac{\cos x}{x \ln x} \right) \)

15. (a) \( \frac{d}{dx} \left( 2x^2(3x^4 - 9x + 7) \right) \)
(b) \( \frac{d}{dx} \left( \sqrt{x} \left( 3x - 2\sqrt{x} + \frac{5}{\sqrt{x}} \right) \right) \)
(c) \( \frac{d}{dx} \left( \frac{2x^3 - 3x^2 + x - 2}{5x^2} \right) \)
(d) \( \frac{d}{dx} \left( \frac{x^3 - 5x + 3}{\sqrt{x}} \right) \)
(e) \( \frac{d}{dx} (\cos x \csc x) \)
(f) \( \frac{d}{dx} \left( \frac{1}{\cot x} + \frac{4 \cos x}{\sin x} \right) \)
(g) \( \frac{d}{dx} \ln(x^{10}) \)
(h) \( \frac{d}{dx} (e^{2\ln x}) \)
(i) \( \frac{d}{dx} \left( 5^{3\log_5 x} \right) \)

16. (a) Find \( f'(x) \) if \( f(x) = \sin x \ln x - \sin (\ln x) \).
(b) Find \( g'(0) \) if \( g(t) = e^{t\tan t} - e^t \tan t \).